SimTech
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Development of a heuristic grid adaptation indicator based on rigorous a posteriori error estimation

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Motivation

Grid adaptation strategies can increase efficiency of simulations. For h-adaptive methods the discretization length h of a cell is changed. Cells are marked for refinement or coarsening based on a local indicator η_i which is compared to a global indicator. These local and global indicators can be heuristic or based on error estimation. Efficient grid adaptation is only guaranteed if error estimators are used which lead to an equal distribution of the error in space and time.

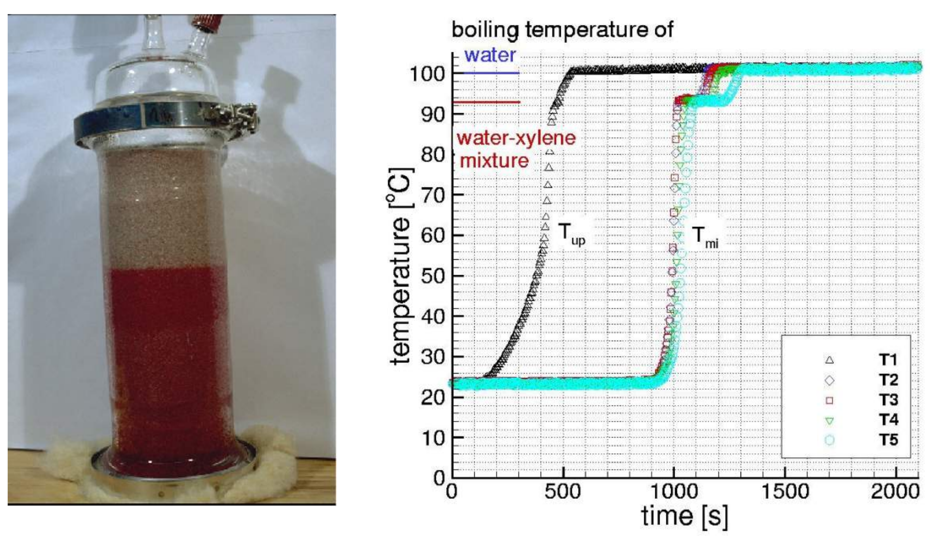

Figure : Column experiment: displacement of trapped xylene by steam injection

Error Estimator

A posteriori error estimator [1]:

$$\int_K |u-u_h| \leq T[a_0 \int_{|x-x_0|<R+1} |u_0(x)-u_h(x,0)|\mathrm{d}x + aQ + 2\sqrt{bcQ}]$$

where

$$\begin{aligned} Q := & \sum_{n\in I_0} \sum_{j\in M(t^n)} |u_j^{n+1}-u_j^n|\Delta t^n h_j^N \\ & + 2\sum_n \sum_{jl\in E(t^n)} \Delta t^n(\Delta t^n + h_{jl}) \max_{u_l^n\leq c\leq d\leq u_j^n} (g_{jl}^n(d,c)-g_{jl}^n(d,d))\delta_{jl}^n \\ & + 2\sum_n \sum_{jl} \Delta t^n(\Delta t^n + h_{jl}) \max_{u_l^n\leq c\leq d\leq u_j^n} (g_{jl}^n(d,c)-g_{jl}^n(c,c))\delta_{jl}^n \\ & + 2M_1 \sum_n \sum_{jl} \Delta t^n h_{jl}^{N-1}(\Delta t^n + h_{jl})^2 \delta_{jl}^n. \end{aligned}$$

- time discretization
- space discretization
- data approximation

Based on this formulation local indicators (η_t, η_x) and global indicators (upper bounds B_t, B_x) can be defined.

Model equations

The error estimator is developed for the following initial value problem for nonlinear conservation laws:

$$\frac{\partial u}{\partial t} + \nabla \cdot F(x,t,u) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+,$$
$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N.$$

Transport equations for the IMPES scheme:

- Immiscible incompressible two phase flow:

$$\frac{\partial S_w \phi}{\partial t} + \nabla \cdot \mathbf{v}_w = 0$$

- Compositional multi phase flow:

$$\frac{\partial C^\kappa}{\partial t} + \nabla \cdot \left(\sum_\alpha X_\alpha^\kappa \varrho_\alpha \mathbf{v}_\alpha \right) = 0$$

Heuristic indicator

Adaptation is governed by the local and global indicators.

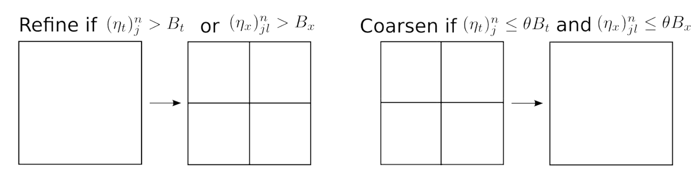

Figure : Adaptation strategy

Dependence of the upper bounds on the number of elements/edges causes fluctuations in refinement of the grid. Additionally the Lipschitz constant in constants *a* and *c* is a very sensitive parameter.

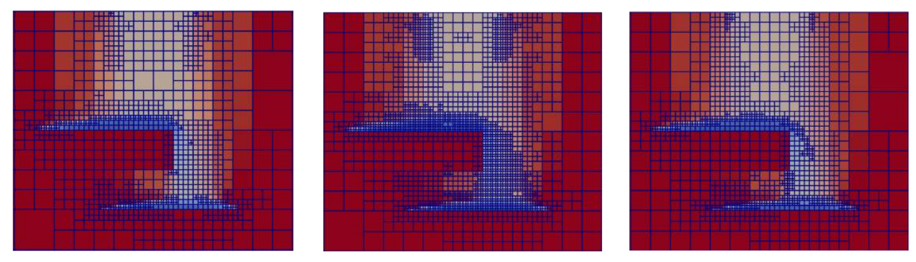

Figure : Fluctuations in grid adaptation for time steps 14, 15, and 16 (simulation of a DNAPL infiltration in heterogeneous domain)

A heuristic indicator is developed by introducing modified upper bounds:

$$B_t = \min\left\{ \alpha\frac{\mathrm{Tol}_t}{\sqrt{a}T^2}, (1-\alpha)^2\frac{\mathrm{Tol}_t^2}{4b\sqrt{c}T^3} \right\},$$
$$B_x = \min\left\{ \beta\frac{\mathrm{Tol}_x}{2\sqrt{a}T^2}, (1-\beta)^2\frac{\mathrm{Tol}_x^2}{8b\sqrt{c}T^3} \right\}.$$

Results

The grid adaptation indicator based on the error estimator leads to a more accurate solution while reducing CPU time.

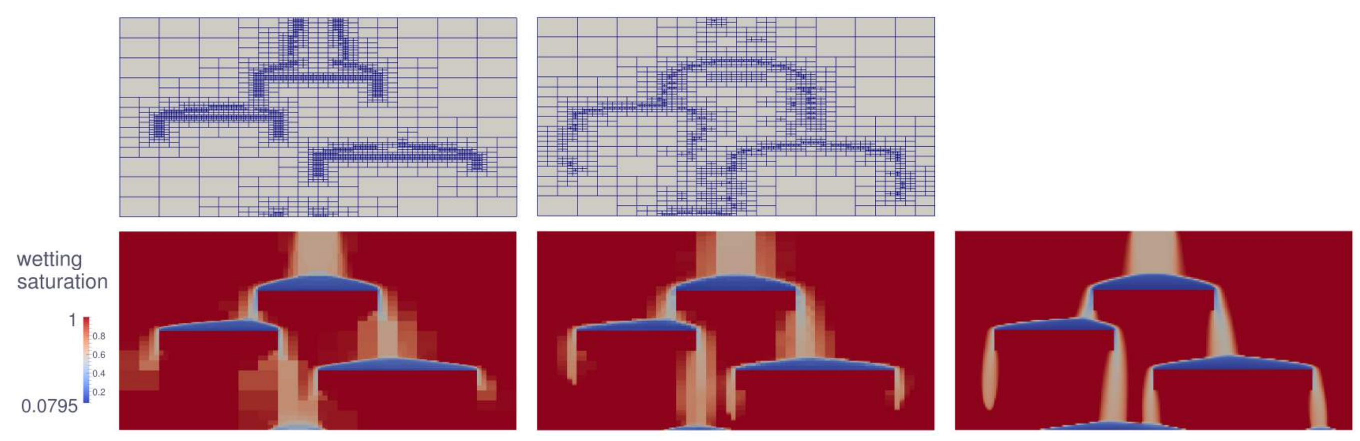

Figure : Grid and saturation distribution for a grid adaptation indicator based on saturation differences (left) and error estimation (middle); reference solution on a fine grid (right).

Outlook

Investigation of applicability in multi-scale [2] and multi-physics models [3].

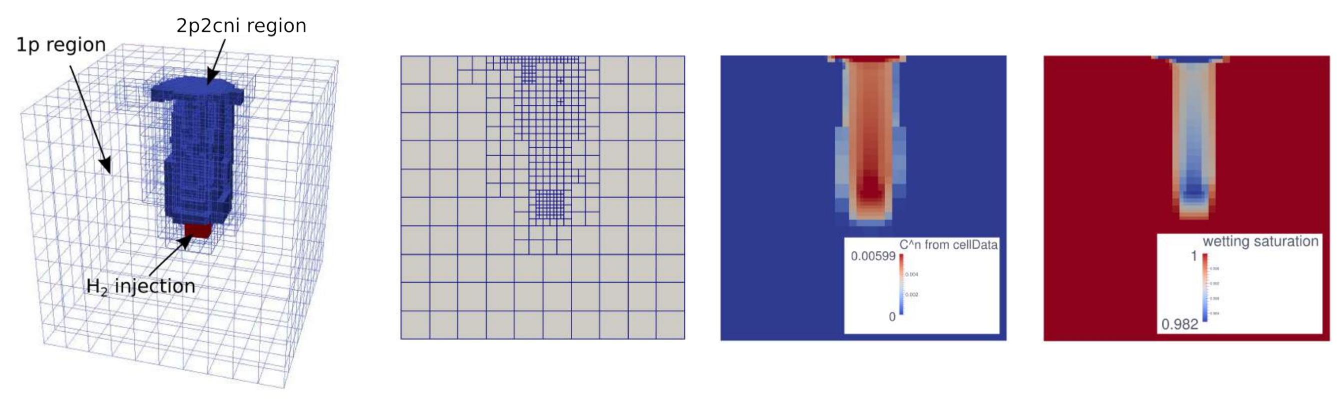

Figure : Simulation of hydrogen injection; from left to right: multi-physics subdomains, refined grid (based on error estimation), total concentration of hydrogen, wetting saturation

References

[1] D. Kröner and M. Ohlberger.
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[2] M. Wolff, B. Flemisch and R. Helmig.
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