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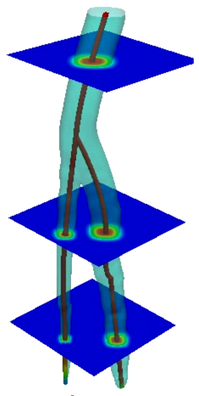
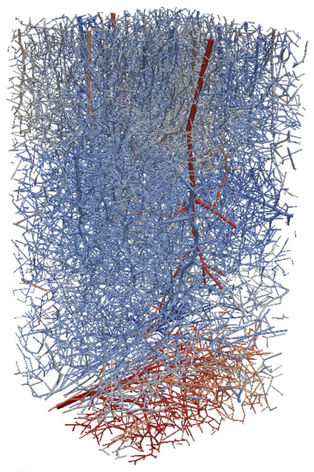
Numerical analysis of elliptic problems with small inclusions

Motivation

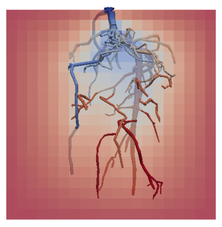
Many biological systems, can be considered as porous media with small inclusions:



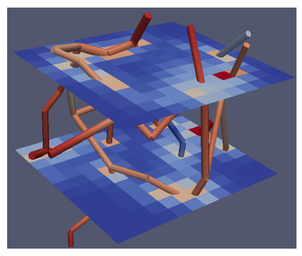
Oxygen transport in muscle tissue



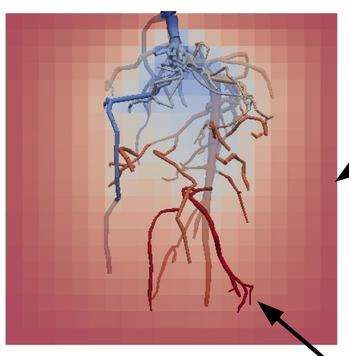
Capillary networks in tissue



Water uptake by roots embedded in soil



Basic modeling approach: Multi-dimension models:



3D Darcy flow models for the porous medium

1D Flow models (Darcy, Poiseuille) for the network structures (inclusion)

3D-1D coupled model

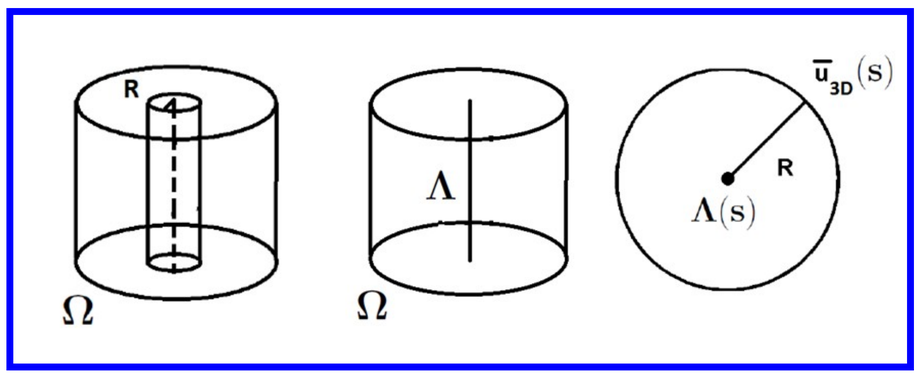
State of current work

Elliptic model problem for the 3D-1D coupling approach:

$$\frac{\partial}{\partial s}\left(k\frac{\partial u_{1D}}{\partial s}\right) = \Phi, \quad \text{in } \Lambda,$$

$$-\nabla\cdot(K\nabla u_{3D}) = \Phi\delta_\Lambda, \quad \text{in } \Omega.$$

Exchange term (network/porous medium): $\Phi = \beta\,(u_{1D} - \overline{u}_{3D})$**.**



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Definitions:

- $\Omega \subset \mathbb{R}^3$: porous matrix, $\Lambda \subset \Omega$: main axis of the inclusions
- u_{3D}: quantity in the 3D porous medium, u_{1D}: quantity in the 1D network
- K: permeabilty of the porous medium, k: permeabilty of the inclusions
- β: permeabilty of the vessel walls, R vessel radius
- Average integral on a circle of radius R around $\Lambda(s)$, perpendicular to Λ:

$$\overline{u}_{3D}(s) = \frac{1}{2\pi}\int_0^{2\pi} u_{3D}\,(\Lambda(s), R, \theta)\ d\theta$$

- Dirac measure δ_Λ:

$$\int_\Omega f\cdot\delta_\Lambda\, dV = \int_\Lambda f\, ds$$

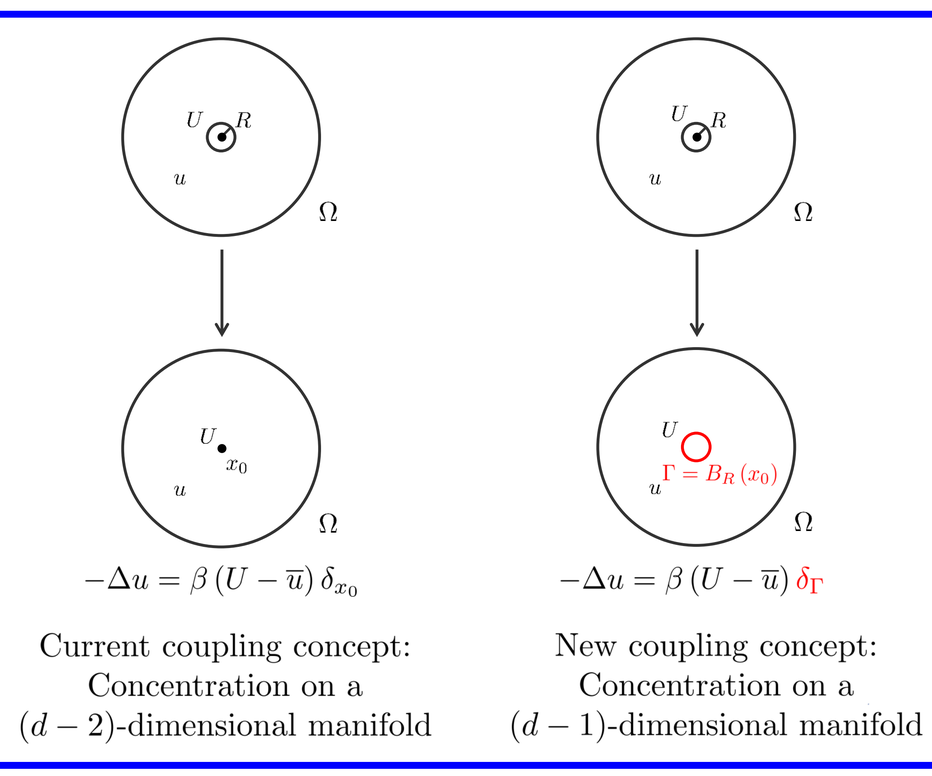
Decoupled problem, straight line Λ**,** $K = I$**,** $U \equiv u_{1D}$**,** $u = u_{3D}$**:**

$$-\Delta u + \beta\,(\overline{u} - U)\,\delta_\Lambda = 0 \ \text{ in } \Omega,\ \ u = u_e \ \text{ on } \partial\Omega.$$

Exact solution $u_e = -c\ln r$**,** r**:** distance to Λ [1, 3].

Problems: Singularity along Λ**, reduced convergence orders.**

New coupling concept: Concentrate the Dirac measure on the surfaces of the inclusions:



Analysis of the new coupling concept [2]:

Higher regularity, global convergence in the H^1**-norm, model reduction errors are bounded by input data.**

Cooperation

This work is a common project with the Institute for Numerical Mathematics, Technical University of Munich (Ettore Vidotto, Barbara Wohlmuth) and the Laboratory for Modeling and Scientific Computing, Polytechnic University of Milan (Paolo Zunino).

References

[1] T. Köppl, E. Vidotto, and B. Wohlmuth.
A local error estimate for the poisson equation with a line source term.
Numerical Mathematics and Advanced Applications ENUMATH 2015, Part(VI):421–430, 2016.

[2] T. Köppl, E. Vidotto, B. Wohlmuth, and P. Zunino.
Model reduction, analysis and numerical approximation of second order elliptic problems with inclusions.
2016, in preparation.

[3] T. Köppl and B. Wohlmuth.
Optimal a priori error estimates for an elliptic problem with dirac right-hand side.
SIAM Journal on Numerical Analysis, 52(4):1753–1769, 2014.

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