Universität Stuttgart

The SIMPLE algorithm -- Solving the Navier-Stokes equation

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SFB Doctoral Workshop
-
25.09.2018

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SIMPLE

Semi-implicit

$$\frac{\varrho^{n+1}\mathbf{v}^{n+1}-\varrho^{n}\mathbf{v}^{n}}{\Delta t}+\nabla\cdot\left(\varrho^{n}\mathbf{v}^{n}\left(\mathbf{v}^{n}\right)^{\mathrm{T}}\right)$$
$$-\nabla\cdot\left(\mu^{n}\left(\nabla\mathbf{v}^{n}+\nabla\left(\mathbf{v}^{n}\right)^{\mathrm{T}}\right)\right)+\nabla p^{n+1}-\varrho^{n}\mathbf{g}-q_{\mathrm{v}}^{n}=0$$

$$\nabla\cdot\mathbf{v}^{n+1}-q_{\mathrm{p}}^{n+1}=0$$

[Laurien et al., Numerische Strömungsmechanik: Grundgleichungen und Modelle, 2018]

SIMPLE	Semi-implicit method for pressure-linked equations

Pressure-linked

$$\nabla\cdot\mathbf{v}-q_{\mathrm{p}}=0$$

$$\frac{\partial(\varrho\mathbf{v})}{\partial t}+\nabla\cdot\left(\varrho\mathbf{v}\mathbf{v}^{\mathrm{T}}\right)-\nabla\cdot\left(\mu\left(\nabla\mathbf{v}+\nabla\mathbf{v}^{\mathrm{T}}\right)\right)+\nabla p-\varrho\mathbf{g}-q_{\mathrm{v}}=0$$

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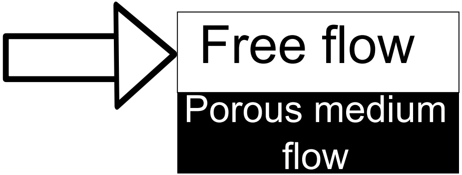
What is it implemented for?



- one component
- isothermal
- laminar
- incompressible

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Linearization of the Navier-Stokes equation

Navier-Stokes equation

$$\frac{\partial(\varrho \mathbf{v})}{\partial t} + \nabla \cdot (\varrho \mathbf{v}\mathbf{v}^{\mathrm{T}}) - \nabla \cdot (\mu(\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathrm{T}})) + \nabla p - \varrho \mathbf{g} - q_{\mathrm{v}} = 0$$

$$\underline{\underline{A}}(\boldsymbol{u})\,\boldsymbol{u} + \underline{\underline{B}}\boldsymbol{p} = \boldsymbol{r_N}$$

Continuity equation

$$\nabla \cdot \mathbf{v} - q_{\mathrm{p}} = 0$$

$$\underline{\underline{C}}\boldsymbol{u} = \boldsymbol{r_C}$$

$$\begin{pmatrix} \boldsymbol{r}_N \\ \boldsymbol{r}_C \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{pmatrix}$$

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The iteration method

$n = 0$:guess
$n \geq 1$: previous value

Step 1
Step 2
Step 3

Solve exactly:

$$\underline{\underline{A}}\left(\boldsymbol{u}^{n-1}\right)\boldsymbol{u}^{n} + \underline{\underline{B}}\boldsymbol{p}^{n} = \boldsymbol{r_N}$$
$$\underline{\underline{C}}\boldsymbol{u}^{n} = \boldsymbol{r_C}$$

Update $\underline{\underline{A}}\left(\boldsymbol{u}\right)$

Stokes equation: $\underline{\underline{A}}$ independent of $\boldsymbol{u}$,
convergence after one iteration step

Navier-Stokes equation:
Iterate until $\underline{\underline{A}}\left(\boldsymbol{u}^{n}\right) = \underline{\underline{A}}\left(\boldsymbol{u}^{n-1}\right)$

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Three steps to an exact solution

Step 1
Step 2
Step 3

Solve exactly:

$$\underline{\underline{A}}\left(\boldsymbol{u}^{n-1}\right)\boldsymbol{u}^n + \underline{\underline{B}}\boldsymbol{p}^n = \boldsymbol{r_N}$$
$$\underline{\underline{C}}\boldsymbol{u}^n = \boldsymbol{r_C}$$

$$\underline{\underline{A}}\delta\tilde{\boldsymbol{u}}^{n_I} = -\boldsymbol{r_N} + \underline{\underline{A}}\boldsymbol{u}^{n_I-1} + \underline{\underline{B}}\boldsymbol{p}^{n_I-1}$$

$$\underline{\underline{C}}\,\underline{\underline{A}}^{-1}\underline{\underline{B}}\delta p = \boldsymbol{r}_C - \underline{\underline{C}}\boldsymbol{u}^{n_I-1} + \underline{\underline{C}}\delta\tilde{\boldsymbol{u}}$$

$$\boldsymbol{p}^{n_I} = \boldsymbol{p}^{n_I-1} - \alpha_p\delta p$$
$$\boldsymbol{u}^{n_I} = \boldsymbol{u}^{n_I-1} - \alpha_u\left(\delta\tilde{\boldsymbol{u}}^{n_I} - \underline{\underline{A}}^{-1}\underline{\underline{B}}\delta p\right)$$

Solve exactly if $\alpha_p = \alpha_u = 1$

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Three steps to an exact solution

Step 1
Step 2
Step 3 includes $\underline{\underline{A}}^{-1}$

Solve exactly:

$$\underline{\underline{A}}\left(\boldsymbol{u}^{n-1}\right)\boldsymbol{u}^{n} + \underline{\underline{B}}\boldsymbol{p}^{n} = \boldsymbol{r_N}$$
$$\underline{\underline{C}}\boldsymbol{u}^{n} = \boldsymbol{r_C}$$

$$\underline{\underline{A}}\delta\tilde{\boldsymbol{u}}^{n_I} = -\boldsymbol{r_N} + \underline{\underline{A}}\boldsymbol{u}^{n_I-1} + \underline{\underline{B}}\boldsymbol{p}^{n_I-1}$$

$$\underline{\underline{C}}\,\underline{\underline{A}}^{-1}\underline{\underline{B}}\delta p = \boldsymbol{r}_C - \underline{\underline{C}}\boldsymbol{u}^{n_I-1} + \underline{\underline{C}}\delta\tilde{\boldsymbol{u}}$$

$$\boldsymbol{p}^{n_I} = \boldsymbol{p}^{n_I-1} - \delta p$$
$$\boldsymbol{u}^{n_I} = \boldsymbol{u}^{n_I-1} - \delta\tilde{\boldsymbol{u}}^{n_I} + \underline{\underline{A}}^{-1}\underline{\underline{B}}\delta p$$

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What SIMPLE does differently

[S.V. Patankar and D.B. Spalding. Int. J. Heat and Mass Transfer, 15:1787, 1972.
Suhas Patankar. Numerical heat transfer and fluid flow. CRC press, 1980.]

Step 1
Step 2
Step 3 includes $\underline{\underline{A}}^{-1}$

Solve ~~exactly~~ approximately:

$$\underline{\underline{A}}\left(\boldsymbol{u}^{n-1}\right)\boldsymbol{u}^{n} + \underline{\underline{B}}\boldsymbol{p}^{n} = \boldsymbol{r}_{N}$$

$$\underline{\underline{C}}\boldsymbol{u}^{n} = \boldsymbol{r}_{C}$$

approximated quite roughly

$$\underline{\underline{A}}^{-1} \approx \left(\mathrm{diag}\underline{\underline{A}}\right)^{-1}$$

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[Suhas Patankar. Numerical heat transfer and fluid flow. CRC press, 1980.]

SIMPLE Revised

$$\hat{u} = -\left(\operatorname{diag} A\right)^{-1} \left((A - \operatorname{diag} A)\right)u - r_N)$$

$$-C\left(\operatorname{diag} A\right)^{-1} Bp^{n_I} = r_C - C\hat{u}$$

$$A\delta\tilde{u} = -r_n + Au^{n_I - 1} + Bp^{n_I}$$

$$C\left(\operatorname{diag} A\right)^{-1} B\delta p = r_C - Cu^{n_I - 1} + C\delta\tilde{u}$$

$$u^{n_I} = u^{n_I - 1} - \delta\tilde{u} + \left(\operatorname{diag} A\right)^{-1} B\delta p$$

Dirichlet boundary conditions

$$\begin{pmatrix} \vdots \\ u_{\text{Dirichlet}} \\ \vdots \\ \boldsymbol{r}_C \end{pmatrix} = \begin{pmatrix} \vdots & \\ 0\ldots010\ldots0 & 0\ldots0 \\ \vdots & \\ \underline{\underline{C}} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{pmatrix}$$

zero line in $\underline{\underline{B}}$

$$\underline{\underline{A}}\delta\tilde{\boldsymbol{u}}^{n_I} = -\boldsymbol{r}_{\boldsymbol{N}} + \underline{\underline{A}}\boldsymbol{u}^{n_I-1} + \underline{\underline{B}}\boldsymbol{p}^{n_I-1}$$

$$\underline{\underline{CA}}^{-1}\underline{\underline{B}}\delta p = \boldsymbol{r}_C - \underline{\underline{C}}\boldsymbol{u}^{n_I-1} + \underline{\underline{C}}\delta\tilde{\boldsymbol{u}}$$

With the details of this sparsity pattern zero line in $\underline{\underline{B}}$ leads to singular matrix $\underline{\underline{CA}}^{-1}\underline{\underline{B}}$

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Fixed pressure in one cell

$$\begin{pmatrix} \boldsymbol{r}_N \\ \vdots \\ p_{\text{Dirichlet}} \\ \vdots \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \vdots & \\ 0\ldots010\ldots0 & 0\ldots0 \\ \vdots & \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{pmatrix}$$

$$\underline{\underline{A}}\delta\tilde{\boldsymbol{u}}^{n_I} = -\boldsymbol{r}_{\boldsymbol{N}} + \underline{\underline{A}}\boldsymbol{u}^{n_I-1} + \underline{\underline{B}}\boldsymbol{p}^{n_I-1}$$

$$\underline{\underline{C}}\,\underline{\underline{A}}^{-1}\underline{\underline{B}}\delta p = \boldsymbol{r}_C - \underline{\underline{C}}\boldsymbol{u}^{n_I-1} + \underline{\underline{C}}\delta\tilde{\boldsymbol{u}}$$

With the details of this sparsity pattern line with one 1 in $\underline{\underline{C}}$ leads to singular matrix $\underline{\underline{C}}\,\underline{\underline{A}}^{-1}\underline{\underline{B}}$ as does not fixing any pressure

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Summary, Results and Outlook

Algorithm to iteratively solve the incompressible Navier-Stokes equation

Convergence generally poor, variants are a bit better

Improve SIMPLE

- implement version without approximation of the inverse
- solving the pressure and the advection equation by different solvers

Combine with other techniques

- preconditioning for other solvers
- warm start before Newton algorithm in DDM context

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Thank you very much for your attention

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