Dimensional Reduction of Field Problems in a Differential-Forms Framework – Extended Version

S. Kurz*, B. Auchmann† and B. Flemisch‡

*ETAS GmbH, Stuttgart, Germany, † CERN, Geneva, Switzerland, ‡IWS, Universität Stuttgart, Germany

E-mail: stefan.kurz@etas.com

***Abstract*: This paper presents a geometric approach to the problem of dimensional reduction. We aim to derive (3+1)-D formulations of 4-D field problems in the relativistic theory of electromagnetism, as well as 2-D formulations of 3-D field problems with continuous symmetries. Our approach is an evolution of recent work [1–4]. It is based solely on calculus with differential forms on differentiable manifolds. The treatment of Schiff's paradox [5] illustrates the potential to analyse complex problems with minimum computational effort and increased conceptual insight.**

***Keywords*: Dimensional reduction, relativistic electrodynamics, numerical formulations.**

I. Introduction

The simulation of electromagnetic field problems involving motion is a challenging subject for engineers and physicists alike. Similarly, the 2-D formulation of field problems in twisted media requires a profound knowledge in vector analysis. Both problems can be stated in terms of a dimensional reduction of a higher-dimensional problem.

The framework for dimensional reduction that we propose is based solely on differential forms on manifolds. This theory of calculus makes a clear distinction between topological and metric concepts, and therefore simplifies the conceptual and computational work. Moreover, a continuous formulation of a field problem with differential forms translates in a straight-forward way into a discrete formulation with discrete fields and operators [6].

Our approach is an evolution of recent work [1–4], striving for a generalization of different approaches, and deliberately avoiding a mix of paradigms (e.g. Ricci calculus and differential forms). The present concept of dimensional reduction is not linked to electromagnetism in Minkowski space alone and can be applied to field problems of arbitrary dimension $n \geq 1$.

In Section II, the projection operators P and Q are introduced, which map p-vectors to their horizontal and transversal components, respectively. Moreover, the splitting operator S is defined, which maps a vector into the pair of its horizontal component and the contracted transversal component. The exterior derivative, the Riesz isomorphism, and the Hodge operator are decomposed w.r.t. a given splitting in Section III.

A so-called observer structure $(\mathbf{u}', \boldsymbol{\mu}^0)$ is discussed in Section IV. The vector field $\mathbf{u}'$ represents a congruence (fibration), and the exact 1-form $\boldsymbol{\mu}^0 = \mathrm{d}\lambda$ constitutes a (codimension-1) foliation, with leafs given by the submanifolds $\lambda = \text{const}$. The vector field distinguishes a direction field, e.g. a 4-velocity field, or a continuous symmetry. The leafs of the foliation define a horizon for the above projections and splittings. In Section V, we apply our framework to electromagnetism in four dimensions and analyze Schiff's paradox as a particular example.

II. Projections and Splittings

A. Multilinear algebra preliminaries

Let $\Lambda^p V$ denote a linear space of p-vectors, with $\Lambda^1 V = V$, and $\Lambda^p V^*$ its dual space. The *generalized contraction*

$$\mathrm{i}\colon \Lambda^q V \times \Lambda^p V^* \to \Lambda^{p-q} V^* : (\mathbf{v}, \boldsymbol{\omega}) \mapsto \mathrm{i}_{\mathbf{v}} \boldsymbol{\omega}, \tag{1}$$

$p \geq q$, is induced from the standard contraction with a 1-vector according to the rule $\mathrm{i}_{\mathbf{w}\wedge\mathbf{u}} = \mathrm{i}_{\mathbf{u}}\,\mathrm{i}_{\mathbf{w}}$ for $\mathbf{u} \in V$, $\mathbf{w} \in \Lambda^{q-1} V$. The *multiplication operator* is defined as

$$\mathrm{j}\colon \Lambda^q V \times \Lambda^p V \to \Lambda^{p+q} V : (\mathbf{u}, \mathbf{v}) \mapsto \mathrm{j}_{\mathbf{u}}\, \mathbf{v} = \mathbf{u} \wedge \mathbf{v}. \tag{2}$$

The operators i and j are defined in the obvious way on the respective dual spaces [7]. In particular, we have

$$\mathrm{i}_{\mathbf{u}} \boldsymbol{\omega} | \mathbf{v} = \boldsymbol{\omega} | \mathrm{j}_{\mathbf{u}}\, \mathbf{v}, \tag{3}$$

where $\cdot\,|\,\cdot$ denotes the covector | vector duality product.

B. Projection of vectors and covectors

Let $\Lambda^p V_{\boldsymbol{\mu}} := \{\mathbf{w} \in \Lambda^p V : \mathrm{i}_{\boldsymbol{\mu}} \mathbf{w} = 0\}$ denote the horizontal subspace of $\Lambda^p V$ w.r.t. $\boldsymbol{\mu} \in V^*$. We introduce the *projection operators*

$$\mathrm{P}_{\mathbf{u}\boldsymbol{\mu}} : \Lambda^p V \to \Lambda^p V_{\boldsymbol{\mu}} : \mathbf{w} \mapsto \frac{1}{\mathrm{i}_{\boldsymbol{\mu}} \mathbf{u}}\, \mathrm{i}_{\boldsymbol{\mu}}\, \mathrm{j}_{\mathbf{u}}\, \mathbf{w}, \tag{4}$$

$$\mathrm{Q}_{\mathbf{u}\boldsymbol{\mu}} : \Lambda^p V \to \Lambda^p V : \mathbf{w} \mapsto \frac{1}{\mathrm{i}_{\boldsymbol{\mu}} \mathbf{u}}\, \mathrm{j}_{\mathbf{u}}\, \mathrm{i}_{\boldsymbol{\mu}}\, \mathbf{w}, \tag{5}$$

for $\mathrm{i}_{\boldsymbol{\mu}} \mathbf{u} = \mathrm{i}_{\mathbf{u}} \boldsymbol{\mu} \neq 0$, which decompose a p-vector into its horizontal and transversal component,

$$\mathbf{w} = \mathrm{P}_{\mathbf{u}\boldsymbol{\mu}} \mathbf{w} + \mathrm{Q}_{\mathbf{u}\boldsymbol{\mu}} \mathbf{w}. \tag{6}$$

Analogously, $\Lambda^p V^*_{\mathbf{u}} := \{\boldsymbol{\omega} \in \Lambda^p V^* : \mathrm{i}_{\mathbf{u}} \boldsymbol{\omega} = 0\}$ denotes the horizontal subspace of $\Lambda^p V^*$ w.r.t. $\mathbf{u} \in V$, and

$$\mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}} : \Lambda^p V^* \to \Lambda^p V^*_{\mathbf{u}} : \boldsymbol{\omega} \mapsto \frac{1}{\mathrm{i}_{\mathbf{u}} \boldsymbol{\mu}}\, \mathrm{i}_{\mathbf{u}}\, \mathrm{j}_{\boldsymbol{\mu}}\, \boldsymbol{\omega}, \tag{7}$$

$$\mathrm{Q}^*_{\mathbf{u}\boldsymbol{\mu}} : \Lambda^p V^* \to \Lambda^p V^* : \boldsymbol{\omega} \mapsto \frac{1}{\mathrm{i}_{\mathbf{u}} \boldsymbol{\mu}}\, \mathrm{j}_{\boldsymbol{\mu}}\, \mathrm{i}_{\mathbf{u}}\, \boldsymbol{\omega}, \tag{8}$$

denote the dual projection operators. The action of the projection operators is illustrated in Fig. 1. The lines $V_{\boldsymbol{\omega},c} := \{\mathbf{w} : \mathrm{i}_{\boldsymbol{\omega}} \mathbf{w} = c\}$ are used to depict a co-vector $\boldsymbol{\omega}$, and the line named $\boldsymbol{\omega}$ corresponds to $V_{\boldsymbol{\omega},1}$, [8]. The geometrical construction of the projected quantities completely avoids the use of any metric property, which nicely emphasizes the fact that this concept is metric-free.

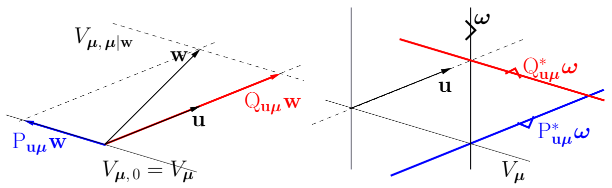

Figure 1. Projecting a vector $\mathbf{w}$ (left) and a covector $\boldsymbol{\omega}$ (right) in a horizontal and a transversal component w.r.t. the pair $(\mathbf{u}, \boldsymbol{\mu})$.

C. Splitting of vectors and covectors

We introduce the *splitting* of covectors w.r.t. $(\mathbf{u}, \boldsymbol{\mu})$,

$$\mathrm{S}^*_{\mathbf{u}\boldsymbol{\mu}} : \Lambda^p V^* \to \Lambda^p V^*_{\mathbf{u}} \times \Lambda^{p-1} V^*_{\mathbf{u}}, \tag{9}$$

and we require that

$$\mathrm{i}_{\mathbf{u}}\, \boldsymbol{\mu} = 1, \tag{10}$$

so that $\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}} + \mathrm{j}_{\boldsymbol{\mu}}\,\mathrm{i}_{\mathbf{u}} = 1$, compare (6). The splitting

$$\mathrm{S}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\omega} := \begin{pmatrix} \mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}} \\ \mathrm{i}_{\mathbf{u}}\, \mathrm{Q}^*_{\mathbf{u}\boldsymbol{\mu}} \end{pmatrix} \boldsymbol{\omega} = \begin{pmatrix} \mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}} \\ \mathrm{i}_{\mathbf{u}} \end{pmatrix} \boldsymbol{\omega} =: \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \tag{11}$$

yields a pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of horizontal covectors of degree p and $(p{-}1)$. Its inverse is given by

$$(\mathrm{S}^*_{\mathbf{u}\boldsymbol{\mu}})^{-1} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{j}_{\boldsymbol{\mu}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}. \tag{12}$$

Analogous definitions apply for the splitting of vectors.

D. Splitting as a structure on a manifold

In what follows, M denotes an n-dimensional, orientable, and non-compact, (pseudo-)Riemannian differentiable manifold, $\mathcal{X}^p(M)$ or $\mathcal{X}^p$ the space of smooth p-vector fields, and $\mathcal{F}^p(M)$ or $\mathcal{F}^p$ differential p-forms. A vector field $\mathbf{u} \in \mathcal{X}^1(M)$ and a 1-form $\boldsymbol{\mu} \in \mathcal{F}^1(M)$, with $\mathrm{i}_{\mathbf{u}}\, \boldsymbol{\mu} = 1$ everywhere, induce projection operators and a splitting in every tangent- and cotangent space of M. We write $\mathrm{S} = (\mathbf{u}, \boldsymbol{\mu})$ for the structure and, in slight abuse of notation, $\mathrm{S} : \mathcal{F}^p \to \mathcal{F}^p_{\mathbf{u}} \times \mathcal{F}^{p-1}_{\mathbf{u}}$ for the action of the splitting on differential forms, where $\mathcal{F}^p_{\mathbf{u}} = \{\boldsymbol{\omega} \in \mathcal{F}^p : \mathrm{i}_{\mathbf{u}}\boldsymbol{\omega} = 0\}$. Splittings are characterized as follows:

- A *metric-compatible* splitting fulfills the relation

$$\boldsymbol{\mu} = \xi^{-2}\mathrm{g}\,\mathbf{u}, \tag{13}$$

  with the Riesz isomorphism g, see Subsection III.B.
- A splitting with

$$\boldsymbol{\mu} = \mathrm{d}\lambda, \tag{14}$$

  i.e., with an exact 1-form $\boldsymbol{\mu}$, is called *holonomic*.
- If the 1-form $\boldsymbol{\mu}$ fulfills

$$\boldsymbol{\mu} \wedge \mathrm{d}\boldsymbol{\mu} = 0 \quad \Leftrightarrow \quad \boldsymbol{\mu} = \alpha\, \mathrm{d}\lambda \tag{15}$$

  for some $\lambda, \alpha \in \mathcal{F}^0$, the splitting is called *integrable*. Holonomic splittings are integrable.

Integrable splittings induce a foliation of M, i.e., a decomposition into $(n{-}1)$-dim. submanifolds $\lambda =$ const. Since $\boldsymbol{\mu}$ defines local horizontal subspaces in the tangent spaces of M, we call S holonomic if these subspaces "connect" to the tangent bundle of an $(n{-}1)$-dim. submanifold, see Fig. 2 (middle). If the connection is a matter of scaling, S is called integrable, see Fig. 2 (right). In general, metric-compatible splittings are not integrable, see Fig. 2 (left).

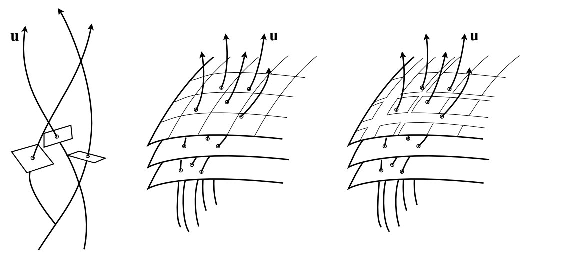

Figure 2. Left: Metric-compatible splitting for a helical congruence. The horizontal subspaces do not match. Middle: Holonomic splitting. The horizontal subspaces connect to form the tangent spaces of the leafs of a foliation. Right: Integrable splitting. The connection of the horizontal subspaces can be achieved by scaling.

III. Decomposition of Operators

A. Exterior derivative

For a given splitting S, we note that $\mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1}\mathrm{S}\,\boldsymbol{\omega}$ is the splitting of $\mathrm{d}\boldsymbol{\omega}$. The action of the exterior derivative in dimension n can therefore be decomposed w.r.t. S. With

$$\hat{\mathrm{d}} := \mathrm{P}_{\mathbf{u}\boldsymbol{\mu}}\,\mathrm{d} = \mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\,\mathrm{d} : \mathcal{F}^p \to \mathcal{F}^{p+1}, \tag{16}$$

the *horizontal exterior derivative* operator, and with the Lie derivative $\mathcal{L}_{\mathbf{u}} = \mathrm{i}_{\mathbf{u}}\mathrm{d} + \mathrm{d}\,\mathrm{i}_{\mathbf{u}} : \mathcal{F}^p \to \mathcal{F}^p$, the decomposition reads

$$\mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1} \;=\; \begin{pmatrix} \hat{\mathrm{d}} & \mathrm{j}_{\boldsymbol{\eta}} \\ \mathcal{L}_{\mathbf{u}} & \mathrm{j}_{\boldsymbol{\delta}} - \hat{\mathrm{d}} \end{pmatrix}, \tag{17}$$

$$\text{where} \quad \boldsymbol{\delta} \;:=\; \mathcal{L}_{\mathbf{u}}\boldsymbol{\mu}, \quad \boldsymbol{\eta} \;:=\; \hat{\mathrm{d}}\boldsymbol{\mu}, \tag{18}$$

are the acceleration 1-form and the vorticity 2-form, respectively [4]. They can be expressed as the splitting of the exterior derivative of $\boldsymbol{\mu}$, $(\boldsymbol{\eta}, \boldsymbol{\delta})^{\mathrm{T}} = \mathrm{S}\,\mathrm{d}\boldsymbol{\mu}$. For a holonomic splitting, $\mathrm{d}\boldsymbol{\mu} = 0$, it follows that $\boldsymbol{\delta} = \boldsymbol{\eta} = 0$, and

$$\mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1} = \begin{pmatrix} \hat{\mathrm{d}} & 0 \\ \mathcal{L}_{\mathbf{u}} & -\hat{\mathrm{d}} \end{pmatrix}. \tag{19}$$

For an integrable splitting, $\boldsymbol{\mu} \wedge \mathrm{d}\boldsymbol{\mu} = 0$, we find equivalently $\boldsymbol{\eta} = 0$. It follows that for $\boldsymbol{\eta} \neq 0$ the 1-form $\boldsymbol{\mu}$ does not represent a foliation of M and that $-\hat{\mathrm{d}}\hat{\mathrm{d}} = \mathrm{j}_{\boldsymbol{\eta}}\mathcal{L}_{\mathbf{u}}$.

For an integrable splitting, however, $\hat{\mathrm{d}}\hat{\mathrm{d}} = 0$, and the Stokes theorem holds on horizontal domains, i.e. on domains D that are subsets of a submanifold $\lambda =$ const. As tangent vectors to D are in $\mathcal{X}^1_{\boldsymbol{\mu}}$ we find for $\boldsymbol{\alpha} \in \mathcal{F}^{p-1}_{\mathbf{u}}$

$$\int_{\partial D} \boldsymbol{\alpha} = \int_D \mathrm{d}\boldsymbol{\alpha} = \int_D \left(\hat{\mathrm{d}} + \mathrm{j}_{\boldsymbol{\mu}}\mathrm{i}_{\mathbf{u}}\mathrm{d}\right)\boldsymbol{\alpha} = \int_D \hat{\mathrm{d}}\boldsymbol{\alpha}. \tag{20}$$

B. Riesz isomorphism

We introduce a metric on M in the form of the Riesz isomorphism

$$\mathrm{g} : \mathcal{X}^1 \to \mathcal{F}^1 : \mathbf{u} \mapsto \mathrm{g}\,\mathbf{u}, \tag{21}$$

where we assume that g gives rise to a symmetric, non-degenerate inner product $\langle \mathbf{u}, \mathbf{w} \rangle = \mathrm{g}\,\mathbf{u}|\mathbf{w}$, and $\langle \boldsymbol{\mu}, \boldsymbol{\nu} \rangle = \boldsymbol{\mu}|\mathrm{g}^{-1}\boldsymbol{\nu}$. We require that $\mathbf{u}$ and $\boldsymbol{\mu}$ fulfill

$$\xi^2 := \langle \mathbf{u}, \mathbf{u} \rangle > 0, \quad \text{and} \quad \eta^2 := \langle \boldsymbol{\mu}, \boldsymbol{\mu} \rangle > 0, \tag{22}$$

which is consistent with our definition of the observer structure in Section IV. The p-compound extends g to p-forms and p-vector fields in the usual way.

We define the *horizontal Riesz isomorphism*

$$\hat{\mathrm{g}} : \mathcal{X}^1_{\mathrm{g}\mathbf{u}}(M) \to \mathcal{F}^1_{\mathbf{u}}(M) : \mathbf{w} \mapsto (-1)^{\sigma_{\mathrm{s}}}\mathrm{g}\mathbf{w}, \tag{23}$$

where σ_{s} allows to change the signature between the global and the horizontal metric. As a consequence of the p-compound construction of the Riesz isomorphism, we find

$$\hat{\mathrm{g}} : \mathcal{X}^p_{\mathrm{g}\mathbf{u}}(M) \to \mathcal{F}^p_{\mathbf{u}}(M) : \mathbf{w} \mapsto (-1)^{p\sigma_{\mathrm{s}}}\mathrm{g}\mathbf{w}. \tag{24}$$

It turns out to be convenient to measure the incompatibilty of the splitting S with the metric by

$$\boldsymbol{\nu} := -\xi^{-1}\mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}\,\mathrm{g}\,\mathbf{u} \;\in \mathcal{F}^1_{\mathbf{u}}. \tag{25}$$

Since $\mathrm{g}\,\mathbf{u} = \mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}\,\mathrm{g}\,\mathbf{u} + \mathrm{Q}^*_{\mathbf{u}\boldsymbol{\mu}}\,\mathrm{g}\,\mathbf{u}$, we have

$$\mathrm{g}\,\mathbf{u} = -\xi\boldsymbol{\nu} + \xi^2\boldsymbol{\mu}, \tag{26}$$

illustrating that metric-compatibility (13) is equivalent to the condition $\boldsymbol{\nu} = 0$. We furthermore set

$$\mathbf{v} := \mathrm{g}^{-1}\boldsymbol{\nu} \;\in \mathcal{X}^1_{\mathrm{g}\mathbf{u}}, \tag{27}$$

and call $\mathbf{v}$ the velocity parameter of S with respect to g. For a physical interpretation, see Table I. It holds that

$$(\xi\eta)^2 = 1 + \mathrm{i}_{\mathbf{v}}\,\boldsymbol{\nu}. \tag{28}$$

C. Hodge operator

Let $\boldsymbol{\Omega} \in \mathcal{F}^n$ denote the volume form on M, $\langle\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle = (-1)^{\sigma_{\mathrm{g}}}$, with σ_{g} the number of negative eigenvalues of the metric. A *horizontal volume form* $\hat{\boldsymbol{\Omega}} \in \mathcal{F}^{n-1}_{\mathbf{u}}$ is defined by $\hat{\boldsymbol{\Omega}}|\hat{\mathrm{g}}^{-1}\hat{\boldsymbol{\Omega}} := (-1)^{\sigma_{\hat{\mathrm{g}}}}$. Provided that $(-1)^{\sigma_{\hat{\mathrm{g}}}} = (-1)^{(n-1)\sigma_{\mathrm{s}}+\sigma_{\mathrm{g}}}$, it reads in explicit form

$$\hat{\boldsymbol{\Omega}} = (-1)^{\sigma_{\mathrm{o}}}\xi^{-1}\,\mathrm{i}_{\mathbf{u}}\boldsymbol{\Omega}, \tag{29}$$

where σ_{o} sets the relative orientation of $\hat{\boldsymbol{\Omega}}$. If V denotes the tangent space in a point of M, then σ_{o} is zero if $\mathrm{Or}(V) = (\mathbf{u}, \mathrm{Or}(V_{\boldsymbol{\mu}}))$ and one otherwise.

The explicit definition of the Hodge operator reads

$$* : \mathcal{F}^p \to \mathcal{F}^{n-p} : \boldsymbol{\alpha} \mapsto (-1)^{\sigma_{\mathrm{g}}}\mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\alpha}}\boldsymbol{\Omega}. \tag{30}$$

The *horizontal Hodge operator* is defined analogously,

$$\hat{*} : \mathcal{F}^p_{\mathbf{u}} \to \mathcal{F}^{n-1-p}_{\mathbf{u}} : \boldsymbol{\alpha} \mapsto (-1)^{\sigma_{\hat{\mathrm{g}}}}\mathrm{i}_{\hat{\mathrm{g}}^{-1}\boldsymbol{\alpha}}\hat{\boldsymbol{\Omega}}, \tag{31}$$

which can be rewritten as

$$\hat{*}\boldsymbol{\alpha} = \xi^{-1}\,\mathrm{n}\,\mathrm{i}_{\mathbf{u}} * \mathrm{m}\,\boldsymbol{\alpha}, \tag{32}$$

where the operators $\mathrm{m}, \mathrm{n} : \mathcal{F}^p \to \mathcal{F}^p$ are defined by

$$\mathrm{m}\,\boldsymbol{\omega} = (-1)^p\boldsymbol{\omega}, \qquad \mathrm{n}\,\boldsymbol{\omega} = (-1)^{p\sigma_{\mathrm{s}}+\sigma_{\mathrm{o}}}\boldsymbol{\omega}. \tag{33}$$

The relations $*\mathrm{j}_{\boldsymbol{\mu}} = \mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\mu}} * \mathrm{m}$ and $\mathrm{j}_{\boldsymbol{\mu}}* = - * \mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\mu}}\mathrm{m}$ are needed to find the decomposition of the Hodge operator

$$\mathrm{S} * \mathrm{S}^{-1} = \mathrm{n}\,\hat{*}\begin{pmatrix} \mathrm{i}_{\mathbf{v}} & \xi\eta^2 - \xi^{-1}\mathrm{j}_{\boldsymbol{\nu}}\,\mathrm{i}_{\mathbf{v}} \\ \xi\mathrm{m} & -\mathrm{j}_{\boldsymbol{\nu}}\,\mathrm{m} \end{pmatrix}. \tag{34}$$

If the splitting is metric-compatible, we have $\boldsymbol{\nu} = 0$, $\mathbf{v} = 0$, and $\xi\eta = 1$, such that the expression simplifies to

$$\mathrm{S} * \mathrm{S}^{-1} = \begin{pmatrix} 0 & \xi^{-1}\mathrm{n}\,\hat{*} \\ \xi\,\mathrm{n}\,\hat{*}\mathrm{m} & 0 \end{pmatrix}. \tag{35}$$

D. Adjoint splitting

The splitting $\mathrm{S} = (\mathbf{u}, \boldsymbol{\mu})$ induces the *adjoint splitting*

$$\widetilde{\mathrm{S}} := (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}}) := (\mathrm{g}^{-1}\boldsymbol{\mu}, \mathrm{g}\,\mathbf{u}). \tag{36}$$

It is obvious that $\mathrm{i}_{\widetilde{\mathbf{u}}}\,\widetilde{\boldsymbol{\mu}} = 1$, that $\widetilde{\widetilde{\mathrm{S}}} = \mathrm{S}$, that $\widetilde{\xi} = \eta$, and that $\widetilde{\eta} = \xi$. We introduce the adjoint velocity parameters $\widetilde{\boldsymbol{\nu}}, \widetilde{\mathbf{v}}$ analogous to (25), (27). In particular, setting

$$\widetilde{\boldsymbol{\nu}} := -\eta^{-1}\mathrm{P}^*_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}}\,\boldsymbol{\mu} \;\in \mathcal{F}^1_{\widetilde{\mathbf{u}}}, \tag{37}$$

we obtain

$$\boldsymbol{\mu} = -\eta\widetilde{\boldsymbol{\nu}} + \eta^2\mathrm{g}\,\mathbf{u}. \tag{38}$$

Moreover, we have that $\widetilde{\mathbf{v}} := \mathrm{g}^{-1}\widetilde{\boldsymbol{\nu}} \in \mathcal{X}^1_{\boldsymbol{\mu}}$ and that $\mathrm{i}_{\widetilde{\mathbf{v}}}\,\widetilde{\boldsymbol{\nu}} = \mathrm{i}_{\mathbf{v}}\,\boldsymbol{\nu}$. Figure 3 illustrates the adjoint splittings S and $\widetilde{\mathrm{S}}$, as well as the relations (26), (38) in terms of their corresponding vector identities.

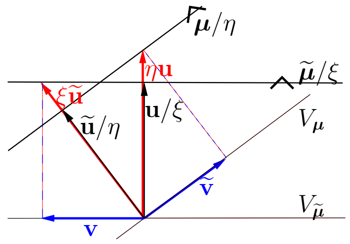

Figure 3. Adjoint splittings S and $\widetilde{\mathrm{S}}$, illustrating that $\xi\widetilde{\mathbf{u}} = \mathbf{v} + \mathbf{u}/\xi$ and that $\eta\mathbf{u} = \widetilde{\mathbf{v}} + \widetilde{\mathbf{u}}/\eta$, corresponding to (26) and (38).

E. Adapted coordinates and modified metric

When performing measurements or numerical calculations, it is at some point inevitable to choose coordinates (x^α), $\alpha = 0, \ldots, n-1$, inducing the canonical bases $B = (\partial x_\alpha) \subset \mathcal{X}^1$ and $B^* = (\mathrm{d}x^\alpha) \subset \mathcal{F}^1$. We say that (x^α) is *adapted* to a holonomic splitting S, if $\mathbf{u} = \partial x_0$ and $\boldsymbol{\mu} = \mathrm{d}x^0$. Then, the horizontal subspaces are spanned by the bases $B_{\boldsymbol{\mu}} = (\partial x_i) \subset \mathcal{X}^1_{\boldsymbol{\mu}}$ and $B^*_{\mathbf{u}} = (\mathrm{d}x^i) \subset \mathcal{F}^1_{\mathbf{u}}$, $i = 1, \ldots, n-1$. As for the adjoint splitting $\widetilde{\mathrm{S}}$ we define a basis $B_{\widetilde{\boldsymbol{\mu}}} = (\mathbf{z}_i)$ of $\mathcal{X}^1_{\widetilde{\boldsymbol{\mu}}}$ by requiring that $\mathrm{d}x^i|\mathbf{z}_j = \delta^i_j$, i.e. $B_{\widetilde{\boldsymbol{\mu}}}$ is dual to $B^*_{\mathbf{u}}$, and a basis $B^*_{\widetilde{\mathbf{u}}} = (\boldsymbol{\zeta}^i)$ of $\mathcal{F}^1_{\widetilde{\mathbf{u}}}$ that is dual to $B_{\boldsymbol{\mu}}$ in the same sense. We find that

$$\partial x_i = \mathrm{P}_{\mathbf{u}\boldsymbol{\mu}}\mathbf{z}_i, \qquad \mathrm{d}x^i = \mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\zeta}^i. \tag{39}$$

For the metric coefficients $g_{ij} := \mathrm{g}\,\partial x_i|\partial x_j$ of g w.r.t. B, and $\hat{g}_{ij} = \hat{\mathrm{g}}\,\mathbf{z}_i|\mathbf{z}_j$ of $\hat{\mathrm{g}}$ w.r.t. $B_{\widetilde{\boldsymbol{\mu}}}$, it holds that

$$\hat{g}_{ij} = (-1)^{\sigma_{\mathrm{s}}}(g_{ij} - \nu_i\nu_j), \tag{40}$$

with $\nu_i = \boldsymbol{\nu}|\mathbf{z}_i$. We remark that (40) corresponds to (7.35) in [9], which is derived there in the language of Ricci calculus and based on the underlying physical context.

For practical calculations, the form (40) of the metric coefficients is inconvenient due to the extra terms $\nu_i\nu_j$. To overcome this disadvantage, we define a *modified metric* $\hat{\mathrm{g}}^\dagger : \mathcal{X}^1_{\widetilde{\boldsymbol{\mu}}} \to \mathcal{F}^1_{\mathbf{u}}$ such that

$$\hat{g}^\dagger_{ij} = (-1)^{\sigma_{\mathrm{s}}}g_{ij}. \tag{41}$$

This can be easily achieved by setting

$$\hat{\mathrm{g}}^\dagger := \mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}\mathrm{P}^*_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}}\,\hat{\mathrm{g}}, \tag{42}$$

yielding indeed

$$\begin{aligned}\hat{g}^\dagger_{ij} &= \hat{\mathrm{g}}^\dagger\,\mathbf{z}_i|\mathbf{z}_j \;=\; (-1)^{\sigma_{\mathrm{s}}}\mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}\mathrm{P}^*_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}}\,\mathrm{g}\,\mathbf{z}_i|\mathbf{z}_j \\ &= (-1)^{\sigma_{\mathrm{s}}}\mathrm{g}\,\mathrm{P}_{\mathbf{u}\boldsymbol{\mu}}\mathbf{z}_i|\mathrm{P}_{\mathbf{u}\boldsymbol{\mu}}\mathbf{z}_j \;=\; (-1)^{\sigma_{\mathrm{s}}}\mathrm{g}\,\partial x_i|\partial x_j \\ &= (-1)^{\sigma_{\mathrm{s}}}g_{ij}.\end{aligned} \tag{43}$$

On $\mathcal{X}^p_{\widetilde{\boldsymbol{\mu}}}$ and $\mathcal{F}^p_{\mathbf{u}}$, we have

$$\hat{\mathrm{g}}^\dagger = (1 + \mathrm{j}_{\boldsymbol{\nu}}\,\mathrm{i}_{\mathbf{v}})\hat{\mathrm{g}}, \text{ and } (\hat{\mathrm{g}}^\dagger)^{-1}(1 + \mathrm{j}_{\boldsymbol{\nu}}\,\mathrm{i}_{\mathbf{v}}) = \hat{\mathrm{g}}^{-1}, \tag{44}$$

respectively. The corresponding *modified volume form* $\hat{\mathbf{\Omega}}^\dagger$ and *modified Hodge operator* $\hat{*}^\dagger$ satisfy $\hat{\mathbf{\Omega}}^\dagger = \xi\eta\hat{\mathbf{\Omega}}$ and

$$\hat{*} = (\xi\eta)^{-1}\hat{*}^\dagger(1 + \mathrm{j}_{\boldsymbol{\nu}}\,\mathrm{i}_{\mathbf{v}}), \tag{45}$$

respectively. Inserting the modified Hodge operator in (34) yields

$$\mathrm{S}*\mathrm{S}^{-1} = \mathrm{n}\hat{*}^\dagger\mathrm{m}\left[\begin{pmatrix} 0 & \eta\mathrm{m} \\ \frac{1}{\eta} & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\xi\eta}\,\mathrm{m}\,\mathrm{i}_{\mathbf{v}} & 0 \\ \frac{1}{\eta}\mathrm{j}_{\boldsymbol{\nu}}\,\mathrm{i}_{\mathbf{v}} & \xi\eta\mathrm{j}_{\boldsymbol{\nu}} \end{pmatrix}\right]. \tag{46}$$

With this equation we can express the Hodge operator, and thus material laws, in terms of the adapted horizontal Hodge operator, plus a perturbation due to the metric-incompatibility of S.

IV. The Observer Structure

A. Definitions and properties

An *observer structure* is given by the pair $(\mathbf{u}', \boldsymbol{\mu}^0)$ of a vector field $\mathbf{u}' \in \mathcal{X}^1$ and an exact 1-form $\boldsymbol{\mu}^0 = \mathrm{d}\lambda \in \mathcal{F}^1$ such that

$$\langle\mathbf{u}', \mathbf{u}'\rangle = \langle\boldsymbol{\mu}^0, \boldsymbol{\mu}^0\rangle = 1. \tag{47}$$

The vector field $\mathbf{u}'$ distinguishes a direction, e.g., a continuous symmetry or a characteristic direction for a material law. The 1-form $\boldsymbol{\mu}^0$ defines an arbitrary foliation on M. It is a convenient tool in computations as it allows, e.g., to give initial values for an initial-value problem on an initial leaf, and to evaluate horizontal fields at well-defined instances $\lambda = $ const.

We set $\gamma := \mathrm{i}_{\mathbf{u}'}\boldsymbol{\mu}^0$. The observer structure enables us to work with three different splittings S', S^0, and S. The characteristic features of these splittings are documented in Table I.

Splitting	Properties
$\mathrm{S}' := (\mathbf{u}',\ \boldsymbol{\mu}' {:=} \mathrm{g}\mathbf{u}')$ $\widetilde{\mathrm{S}'} = \mathrm{S}'$	• anholonomic (in general) • metric-compatible • "threading observer", [3]
$\mathrm{S}^0 := (\mathbf{u}^0 {:=} \mathrm{g}^{-1}\boldsymbol{\mu}^0,\ \boldsymbol{\mu}^0)$ $\widetilde{\mathrm{S}^0} = \mathrm{S}^0$	• holonomic • metric-compatible • "slicing observer", [3]
$\mathrm{S} := (\mathbf{u} {:=} \mathbf{u}'/\gamma,\ \boldsymbol{\mu} {:=} \boldsymbol{\mu}^0)$ $\widetilde{\mathrm{S}} = (\widetilde{\mathbf{u}} {=} \mathbf{u}^0,\ \widetilde{\boldsymbol{\mu}} {=} \boldsymbol{\mu}'/\gamma)$	• holonomic • metric-incompatible (i.g.) • $\mathbf{v} \in \mathcal{X}^1_{\boldsymbol{\mu}'}$ velocity of S' relative to S^0 • $\mathbf{u}^0 = \gamma(\mathbf{v} + \mathbf{u}')$, see Fig. 3 • $\widetilde{\mathbf{v}} \in \mathcal{X}^1_{\boldsymbol{\mu}^0}$ velocity of S^0 relative to S' • $\mathbf{u}' = \gamma(\widetilde{\mathbf{v}} + \mathbf{u}^0)$, see Fig. 3

Table I: Three splittings associated with an observer structure $(\mathbf{u}', \boldsymbol{\mu}^0)$.

The decomposed Hodge operator w.r.t. S is given by (34) or (46), respectively, together with $\xi = \gamma^{-1}$ and $\eta = 1$. Note that the horizontal Hodge operator $\hat{*} : \mathcal{F}^p_{\mathbf{u}} \to \mathcal{F}^{n-1-p}_{\mathbf{u}}$ is defined identically in the splittings S and S'.

B. Different decompositions of field problems

With the given tools we have the choice between two decompositions of the 2nd order equation $\mathrm{d}*\mathrm{d}\boldsymbol{\omega} = \boldsymbol{\rho}$:

1) We can decompose it w.r.t. S'

$$\mathrm{S}'\,\mathrm{d}(\mathrm{S}')^{-1}\,\mathrm{S}'*(\mathrm{S}')^{-1}\,\mathrm{S}'\,\mathrm{d}(\mathrm{S}')^{-1}\,\mathrm{S}'\,\boldsymbol{\omega} = \mathrm{S}'\,\boldsymbol{\rho}. \tag{48}$$

   This approach has the clear advantage of being metric-compatible. However, it is generally anholonomic, and thus the Stokes theorem does not hold, which results in a very limited usability for numerical field calculations.

2) Decomposing w.r.t. S yields

$$\mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1}\,\mathrm{S}*\mathrm{S}^{-1}\,\mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1}\,\mathrm{S}\,\boldsymbol{\omega} = \mathrm{S}\,\boldsymbol{\rho}, \tag{49}$$

   with a complicated decomposed Hodge operator.

The representation of the decomposed Hodge operator in a metric-incompatible splitting can also be cast into the form $\mathrm{S}*\mathrm{S}^{-1} = \mathrm{S}(\mathrm{S}')^{-1}\,\mathrm{S}'*(\mathrm{S}')^{-1}\,\mathrm{S}'\,\mathrm{S}^{-1}$, with the metric-compatible decomposition of the Hodge (35) in the middle of the expression, and with

$$\mathrm{S}(\mathrm{S}')^{-1} = \begin{pmatrix} 1 & -\mathrm{j}_{\boldsymbol{\nu}} \\ 0 & \gamma^{-1} \end{pmatrix}, \quad \mathrm{S}'\,\mathrm{S}^{-1} = \begin{pmatrix} 1 & \gamma\,\mathrm{j}_{\boldsymbol{\nu}} \\ 0 & \gamma \end{pmatrix}. \tag{50}$$

Note that we cannot simply use the metric-compatible and holonomic splitting $\mathrm{S}^0 = (\mathbf{u}^0, \boldsymbol{\mu}^0)$. The vector field $\mathbf{u}'$ distinguishes a direction among all others. Using S^0, we would lose that information.

If the observer structure $(\mathbf{u}', \boldsymbol{\mu}^0)$ obeys $\boldsymbol{\mu}^0 = \mathrm{g}\mathbf{u}'$, then $\mathrm{S}' = \mathrm{S} = \mathrm{S}^0$ and all approaches coincide. With the simple relations (19) and (35), the decomposition of the 2nd order equation then yields two sets of equations that are coupled only by the Lie derivative $\mathcal{L}_{\mathbf{u}'}$.

V. Electromagnetism in Spacetime

A. Physical and mathematical setting

In relativistic electrodynamics, the manifold M is a four-dimensional pseudo-Riemannian manifold with a metric of signature $(+\,-\,-\,-)$, referred to as Minkowski space. The vector-field and 1-form of an observer structure $(\mathbf{u}', \boldsymbol{\mu}^0)$ have unit norm by definition, $\langle\mathbf{u}', \mathbf{u}'\rangle = \langle\boldsymbol{\mu}^0, \boldsymbol{\mu}^0\rangle = 1$. Vectors and covectors with positive square-norm are called *time-like*, those with negative square-norm *space-like*.

The vector field $\mathbf{u}'$ is related to the *four-velocity* by the inverse of the vacuum speed of light $c_0 = (\varepsilon_0\mu_0)^{-1/2}$, where μ_0 and ε_0 denote the magnetic and electric permeability of empty space, respectively. The integral curves of $\mathbf{u}'$ are called *world-lines* and every worldline corresponds to one *point in space*. The exact 1-form $\boldsymbol{\mu}^0 = \mathrm{d}\tau$ is called a *time-synchronization*. Each submanifold $\tau = $ const. corresponds to one *point in time*. The scalar τ is related to *synchronous time* t via $\tau = c_0 t$. *Relative time* is measured along the world-lines of $\mathbf{u}'$ by the metric. *Relative space* is given by the local, horizontal subspaces w.r.t. $\boldsymbol{\mu}' = \mathrm{g}\,\mathbf{u}'$. According to the *hypothesis of locality* [10], the splitting S' defines the 3-D measurable fields on the worldlines $\mathbf{u}'$.

If the covariant derivative of $\mathbf{u}'$ vanishes, $D\mathbf{u}' = 0$, the integral curves are parallel straight lines. Then, $\mathrm{d}\,\mathrm{g}\,\mathbf{u}' = 0$, and thus $\mathrm{g}\,\mathbf{u}' = \mathrm{d}\,\tau$, corresponding to the *Einstein synchronization*, and the observer structure represents a *standard inertial observer*, [3].

With the definitions (25) and (27) of $\mathbf{v}$ and $\boldsymbol{\nu}$, we set

$$\beta^2 \quad := \quad -\mathrm{i}_{\mathbf{v}}\boldsymbol{\nu} = -\mathrm{i}_{\widetilde{\mathbf{v}}}\widetilde{\boldsymbol{\nu}}, \tag{51}$$

obtain from (28) with $\xi^{-1} = \gamma$ and $\eta = 1$ that

$$\gamma \quad = \quad (1-\beta^2)^{-1/2}, \tag{52}$$

and observe that βc_0 is the *relative velocity* between S' and S^{o}, and that γ is the *Lorentz factor*. We choose a positive-definite horizontal metric with the standard right-hand screw orientation, i.e., $\sigma_{\mathrm{s}} = \sigma_{\mathrm{o}} = 1$.

B. Splitting Maxwell and constitutive equations

The Maxwell equations on M read

$$\mathrm{d}F = 0, \quad \mathrm{d}G = J. \tag{53}$$

The quantities are split into the conventional (3+1) electromagnetic field quantities [1], e.g., for the splitting S

$$\begin{pmatrix} D \\ H/c_0 \end{pmatrix} = \mathrm{S}\,G, \; \begin{pmatrix} B \\ -E/c_0 \end{pmatrix} = \mathrm{S}\,F, \; \begin{pmatrix} \rho \\ -j/c_0 \end{pmatrix} = \mathrm{S}\,J. \tag{54}$$

The (3+1) Maxwell equations are obtained with (19) in their standard form from

$$\mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1}\,\mathrm{S}\,F = 0, \qquad \mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1}\,\mathrm{S}\,G = \mathrm{S}\,J. \tag{55}$$

The constitutive law on M in empty space reads

$$G = Z_0^{-1} * F, \tag{56}$$

with $Z_0 = \sqrt{\mu_0 \varepsilon_0^{-1}}$. Due to $** = (-1)^{\sigma_{\mathrm{g}} + p(n-p)} = -1$ with $p = 2$, the inverse of the constitutive law reads $F = -Z_0 * G$. We use (46) to decompose the constitutive law to obtain

$$D = \varepsilon_0 \hat{*}^\dagger E - \gamma Z_0^{-1} \hat{*}^\dagger\, \mathrm{i}_{\mathbf{v}} B = \varepsilon_0 \hat{*}^\dagger E + P_{\mathrm{S}}, \tag{57}$$

$$B = \mu_0 \hat{*}^\dagger H + \gamma Z_0 \hat{*}^\dagger\, \mathrm{i}_{\mathbf{v}} D \;= \mu_0 \hat{*}^\dagger (H + M_{\mathrm{S}}), \tag{58}$$

defining P_{S} and M_{S}. Thus, in the splitting S, empty space generally has a nonzero polarization and magnetization. We introduce the field quantities

$$\bar{D} := \quad \varepsilon_0 \hat{*}^\dagger E \quad = D - P_{\mathrm{S}}, \tag{59}$$

$$\bar{H} := \quad \mu_0^{-1} \hat{*}^\dagger B \quad = H + M_{\mathrm{S}}, \tag{60}$$

and define

$$\hat{\mathrm{d}}\bar{H} = j + j_{\mathrm{S}} + \mathcal{L}_{\mathbf{u}}\bar{D}, \quad j_{\mathrm{S}} := \hat{\mathrm{d}} M_{\mathrm{S}} + \mathcal{L}_{\mathbf{u}} P_{\mathrm{S}}, \tag{61}$$

$$\hat{\mathrm{d}}\bar{D} = \rho + \rho_{\mathrm{S}}, \qquad \rho_{\mathrm{S}} := -\hat{\mathrm{d}} P_{\mathrm{S}}, \tag{62}$$

with the Schiff charge- and current-density ρ_{S} and j_{S}.

C. Schiff's paradox

We apply our framework to Schiff's paradox [5], and quote Schiff's description: "... Consider two concentric spheres with equal and opposite total charges uniformly distributed over their surfaces. When the spheres are at rest, the electric and magnetic fields outside the spheres vanish. ... then an observer traveling in a circular orbit around the spheres should find no field, for since all of the components of the electromagnetic field tensor vanish in one coordinate system, they must vanish in all coordinate systems. On the other hand, the spheres are rotating w.r.t. this observer, and so he should experience a magnetic field." A classical treatment can be found in [11, 12].

We construct an observer structure $(\mathbf{u}', \boldsymbol{\mu}^{\mathrm{o}})$ such that Schiff's first observer experiences the field problem in the holonomic, metric-compatible splitting S^{o}, and the second one makes measurements in the metric-compatible splitting S'. We know that the 4-D field problem is symmetric w.r.t. $\mathbf{u}^{\mathrm{o}}$, i.e., $\mathcal{L}_{\mathbf{u}^{\mathrm{o}}} = 0$, and that $\mathrm{i}_{\mathbf{u}^{\mathrm{o}}} J = 0$, i.e., it is static for observer one. The field $c_0 \widetilde{\mathbf{v}}$ describes the velocity vector-field of a rigid rotation around a particular world-line of the field $\mathbf{u}^{\mathrm{o}}$, the axis of rotation. The world-lines of the rotating observer are given by $\mathbf{u}' = \gamma(\widetilde{\mathbf{v}} + \mathbf{u}^{\mathrm{o}})$. We express the problem in the splitting S to calculate the fields experienced by the rotating observer. Since the problem of observer one is axisymmetric, i.e., $\mathcal{L}_{\widetilde{\mathbf{v}}} = 0$, we have

$$\mathcal{L}_{\mathbf{u}} = \mathcal{L}_{\widetilde{\mathbf{v}} + \mathbf{u}^{\mathrm{o}}} = \mathcal{L}_{\widetilde{\mathbf{v}}} + \mathcal{L}_{\mathbf{u}^{\mathrm{o}}} = 0. \tag{63}$$

Therefore, (61) becomes

$$\hat{\mathrm{d}}\bar{H} = j + j_{\mathrm{S}}, \tag{64}$$

where j is the convective current density in $\mathrm{S}\,J = (\rho, -j/c_0)^{\mathrm{T}}$. With $\mathbf{u}^{\mathrm{o}} = \mathbf{u} - \widetilde{\mathbf{v}}$, it follows that $\mathrm{S}\,\mathbf{u}^{\mathrm{o}} = (-\widetilde{\mathbf{v}}, 1)^{\mathrm{T}}$ with $\widetilde{\mathbf{v}} \in \mathcal{X}^1_{\boldsymbol{\mu}}$. By decomposing $\mathrm{i}_{\mathbf{u}^{\mathrm{o}}} J = 0$ w.r.t. S, we arrive at

$$j = -c_0 \mathrm{i}_{\widetilde{\mathbf{v}}}\, \rho. \tag{65}$$

At the same time the Schiff current-density reads

$$\begin{aligned} j_{\mathrm{S}} \quad &= \quad \hat{\mathrm{d}} M_{\mathrm{S}} = c_0 \hat{\mathrm{d}} \gamma\, \mathrm{i}_{\mathbf{v}} D \\ &= \quad -c_0 \hat{\mathcal{L}}_{\widetilde{\mathbf{v}}} D + c_0 \mathrm{i}_{\widetilde{\mathbf{v}}}\, \hat{\mathrm{d}} D = c_0 \mathrm{i}_{\widetilde{\mathbf{v}}}\, \rho. \end{aligned} \tag{66}$$

We use that $\gamma\, \mathrm{i}_{\mathbf{v}} = -\mathrm{i}_{\widetilde{\mathbf{v}}}$ on $\mathcal{F}^p_{\mathbf{u}}$. From $\mathcal{L}_{\widetilde{\mathbf{v}}} = 0$ on $\mathcal{F}^p$, it follows that $\hat{\mathcal{L}}_{\widetilde{\mathbf{v}}} = 0$ on $\mathcal{F}^p_{\mathbf{u}}$ for the horizontal Lie derivative in the splitting S, $\hat{\mathcal{L}}_{\mathbf{a}} = \mathrm{i}_{\mathbf{a}} \hat{\mathrm{d}} + \hat{\mathrm{d}}\, \mathrm{i}_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{X}^1_{\boldsymbol{\mu}}$.

The Schiff currents cancel out with the convective currents in (64). The boundary condition of finite field at infinity fixes the magnetic field measured by the rotating observer to zero. We are left with

$$\hat{\mathrm{d}}\bar{D} = \rho + \rho_{\mathrm{S}}, \tag{67}$$

where from the above result it follows that $B = 0$ and thus

$$\rho_{\mathrm{S}} = -\hat{\mathrm{d}} P_{\mathrm{S}} = 0. \tag{68}$$

Consequently $D = \bar{D} = \varepsilon_0 \hat{*}^\dagger E$ and the field problem can be solved in the splitting S by solving the Poisson equation

$$-\hat{\mathrm{d}} \varepsilon_0 \hat{*}^\dagger \hat{\mathrm{d}} \varphi = \rho. \tag{69}$$

By introducing adapted coordinates according to Section III.E, a coordinate expression of (69) can be derived. Thanks to (41), this coordinate expression does not depend at all on the velocity parameters. It must therefore be identical to the case $\mathbf{v} = 0$, which coincides with Schiff's first observer. We conclude that the electrical field and thus all fields have to vanish outside the spheres in the splitting S. Now, (50) indicates that this also has to hold for the measurable fields, i.e. the second observer, which resolves the paradox.

VI. Electromagnetism in three dimensions

We discuss the splitting of 3-D field problems with continuous symmetries into 2-D problems, assuming that we can find a vector field $\mathbf{u}$ such that neither fields, nor material properties, nor metric vary along the flow of $\mathbf{u}$, i.e. $\mathcal{L}_{\mathbf{u}} = 0$. As a consequence, for metric-compatible and holonomic splittings, the (3+1) Maxwell equations and constitutive laws decompose into two sets, the transverse electric (TE) and the transverse magnetic (TM) equations.

A. Translational symmetry

In a Cartesian coordinate system (x, y, z) we use the splitting $(\partial_z, \mathrm{d}z)$, $\xi = 1$, which is metric compatible and holonomic, i.e., $\mathrm{S} = \mathrm{S}^{\mathrm{o}} = \mathrm{S}'$, and find the parallel and orthogonal electromagnetic quantities $(E_{\|},\, E_{\perp})^{\mathrm{T}} := \mathrm{S}\, E$ and analogously for H, $(B_{\perp},\, B_{\|})^{\mathrm{T}} := \mathrm{S}\, B$ and analogously for D and j, and $(0,\, \rho_{\perp})^{\mathrm{T}} := \mathrm{S}\, \rho$. We assume translational symmetry along ∂_z. The TE and TM systems read

$$\begin{array}{ll} \text{TE-system} & \text{TM-system} \\ -\hat{\mathrm{d}}\, E_{\perp} = -\partial_t B_{\|}, & \hat{\mathrm{d}}\, E_{\|} = -\partial_t B_{\perp}, \\ -\hat{\mathrm{d}} B_{\|} = 0, & \hat{\mathrm{d}} B_{\perp} = 0, \\ \hat{\mathrm{d}} H_{\|} = j_{\perp} + \partial_t D_{\perp}, & -\hat{\mathrm{d}} H_{\perp} = j_{\|} + \partial_t D_{\|}, \\ \hat{\mathrm{d}} D_{\perp} = 0, & -\hat{\mathrm{d}} D_{\|} = \rho_{\perp}. \end{array} \tag{70}$$

With $\sigma_{\mathrm{s}} = \sigma_{\mathrm{o}} = \sigma_{\hat{\mathrm{g}}} = \sigma_{\mathrm{g}} = 0$, we find $\mathrm{n} = 1$ and

$$\begin{array}{ll} \mu_0^{-1} \hat{*} B_{\|} = H_{\|}, & \mu_0^{-1} \hat{*} B_{\perp} = H_{\perp}, \\ \varepsilon_0 \hat{*} E_{\perp} = D_{\perp}, & -\varepsilon_0 \hat{*} E_{\|} = D_{\|}. \end{array} \tag{71}$$

B. Helical symmetry

We consider a magnetostatic field problem in empty space with helical symmetry. All currents are directed along helical paths that are the integral curves of $\mathbf{u}$, i.e., $\mathrm{i}_{\mathbf{u}} j = 0$. Helical coordinates (R, ϕ, Z) are derived from cylindrical coordinates (r, φ, z) via $\phi = \varphi - \alpha z$ for constant α, $R = r$, and $Z = z$. A 2-D formulation on $z =$ const. suitable for numerical simulation is found by using the modified Hodge operator (46) in the splitting $\mathrm{S} = (\partial_Z, \mathrm{d}Z)$, resulting in

$$H_{\|} = \mu_0^{-1} \hat{*}^{\dagger} B_{\|} - \gamma^{-1} \mathrm{j}_{\boldsymbol{\nu}} \bar{H}_{\perp} =: \bar{H}_{\|} - M_{\mathrm{s}\|}, \tag{72}$$
$$H_{\perp} = \mu_0^{-1} \hat{*}^{\dagger} B_{\perp} + \mathrm{i}_{\tilde{\mathbf{v}}} H_{\|} =: \bar{H}_{\perp} - M_{\mathrm{s}\perp}. \tag{73}$$

Since $\mathrm{i}_{\mathbf{u}'} j = 0$, we find that $j_{\|} = 0$. From (19) we obtain the coupled equations that define the 2-D field problem

$$\hat{\mathrm{d}} \bar{H}_{\|} \;=\; j_{\perp} + j_{\mathrm{s}\perp}, \qquad -\hat{\mathrm{d}} \bar{H}_{\perp} \;=\; j_{\mathrm{s}\|}, \tag{74}$$

where $j_{\mathrm{s}\perp} = \hat{\mathrm{d}} M_{\mathrm{s}\|} = \hat{\mathrm{d}} \gamma^{-1} \mathrm{j}_{\boldsymbol{\nu}} \bar{H}_{\perp}$ and $j_{\mathrm{s}\|} = -\hat{\mathrm{d}} M_{\mathrm{s}\perp} = \hat{\mathrm{d}}\, \mathrm{i}_{\tilde{\mathbf{v}}} H_{\|}$. Note that as the pitch α of the helix decreases, the metric incompatibility of S becomes smaller, so that $\tilde{\mathbf{v}} = \alpha \partial\varphi \to 0$ for $\alpha \to 0$. For straight currents we are left with the magnetostatic problem $\hat{\mathrm{d}} \bar{H}_{\|} = j_{\perp}$ and $\bar{H}_{\|} = H_{\|}$. A classical treatment can be found, e.g., in [13].

VII. Conclusion

We have presented a formalism for the dimensional reduction of field problems in a differential-form framework. The concept of an observer structure and its three splittings S', S^{o}, and S generalizes various approaches in relativistic electrodynamics: [1] uses only metric-compatible, holonomic splittings, whereas in [2, 4] holonomic and anholonomic metric-compatible splittings are employed. The classic book of van Bladel [9] focuses on Ricci calculus, which often makes it difficult to see in what subspaces the relevant quantities are defined. The lack of a clear separation between metric- and topological concepts reduces the conceptual insight. [3] is conceptually very close to this paper. The mix of Ricci-calculus, vector-calculus and differential forms makes the text more involved, though.

We have introduced an adapted metric that allows to write the decomposed Hodge operator in the metric-incompatible splitting S as the sum of a metric-compatible decomposition and a perturbation. In the electromagnetic theory, this perturbation can be treated like a magnetization and polarization of empty space. With this concept, we are able to solve Schiff's paradox without use of coordinates. The differential-form approach clearly favours to work conceptually, rather than technically. Moreover, the differential-form framework for dimensional reduction can be applied to any field problem of dimension $n \geq 1$.

Beyond the scope of this paper, and thus subject of our ongoing work, are various topics of electrodynamics, such as the decomposition of traces, jump- and boundary-conditions for moving domains, the local and global form of Lorentz transformation, the impact of an alternate choice of a time-synchronization, and material-laws in moving bodies. The present formalism can be used to generate concise formulations of complex field problems, involving either moving domains, or a dimensional reduction of field problems with continuous symmetries, e.g., helical symmetry in 3-D. The differential-form formulation can be readily translated into the language of discrete fields and operators [6], and is thus amenable to numerical field calculation.

Appendix

Projections and Splittings

Multilinear algebra preliminaries: The following rule is used extensively in the derivations of this appendix

$$\mathrm{i}_{\boldsymbol{\mu}} \mathrm{j}_{\mathbf{u}} = \mathrm{i}_{\boldsymbol{\mu}} \mathbf{u} - \mathrm{j}_{\boldsymbol{\mu}} \mathrm{i}_{\mathbf{u}} \tag{75}$$

and hence

$$\mathrm{i}_{\boldsymbol{\mu}} \mathrm{j}_{\mathbf{u}} = 1 - \mathrm{j}_{\boldsymbol{\mu}} \mathrm{i}_{\mathbf{u}} \quad \text{for} \quad \mathrm{i}_{\boldsymbol{\mu}} \mathbf{u} = 1. \tag{76}$$

Projection of vectors and covectors: With (75), we show (6) by

$$\begin{aligned} (\mathrm{P}_{\mathbf{u}\boldsymbol{\mu}} + \mathrm{Q}_{\mathbf{u}\boldsymbol{\mu}})\, \mathbf{w} &= \frac{1}{\mathrm{i}_{\boldsymbol{\mu}} \mathbf{u}} (\mathrm{i}_{\boldsymbol{\mu}} \mathrm{j}_{\mathbf{u}} + \mathrm{j}_{\mathbf{u}} \mathrm{i}_{\boldsymbol{\mu}})\, \mathbf{w} \\ &= \frac{1}{\mathrm{i}_{\boldsymbol{\mu}} \mathbf{u}} (\mathrm{i}_{\boldsymbol{\mu}} \mathbf{u} - \mathrm{j}_{\mathbf{u}} \mathrm{i}_{\boldsymbol{\mu}} + \mathrm{j}_{\mathbf{u}} \mathrm{i}_{\boldsymbol{\mu}})\, \mathbf{w} \\ &= \mathbf{w}, \end{aligned} \tag{77}$$

and similarly for the dual operators $\mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}$ and $\mathrm{Q}^*_{\mathbf{u}\boldsymbol{\mu}}$.

Visualization of the projections: The action of the projection operators is illustrated in Fig. 1. The transversal component $\mathrm{Q}_{\mathbf{u}\boldsymbol{\mu}} \mathbf{w}$ of the vector $\mathbf{w}$ is obtained by scaling $\mathbf{u}$ such that $\mathrm{i}_{\boldsymbol{\mu}} \mathrm{Q}_{\mathbf{u}\boldsymbol{\mu}} \mathbf{w} = \mathrm{i}_{\boldsymbol{\mu}} \mathbf{w}$, while the horizontal component $\mathrm{P}_{\mathbf{u}\boldsymbol{\mu}} \mathbf{w}$ can be constructed by the requirements that $\mathrm{P}_{\mathbf{u}\boldsymbol{\mu}} \mathbf{w} \in V_{\boldsymbol{\mu}}$, and that the 2-vectors $\mathrm{j}_{\mathbf{u}} \mathrm{P}_{\mathbf{u}\boldsymbol{\mu}} \mathbf{w}$ and

$\mathrm{j}_{\mathbf{u}}\mathbf{w}$ have the same orientation and area. Analogously and simply by exchanging the role of vectors and covectors, the transversal component $\mathrm{Q}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\omega}$ of the covector $\boldsymbol{\omega}$ is obtained by scaling $\boldsymbol{\mu}$ such that $\mathrm{i}_{\mathbf{u}}\,\mathrm{Q}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\omega} = \mathrm{i}_{\mathbf{u}}\boldsymbol{\omega}$, while the horizontal component $\mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\omega}$ can be constructed by the requirements that $\mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\omega} \in V^*_{\mathbf{u}}$, and that the 2-covectors $\mathrm{j}_{\boldsymbol{\mu}}\mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\omega}$ and $\mathrm{j}_{\boldsymbol{\mu}}\boldsymbol{\omega}$ have the same orientation and area.

Splitting of vectors and covectors: We show (11) by

$$\begin{aligned}\mathrm{i}_{\mathbf{u}}\,\mathrm{Q}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\omega} &= \mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\mathrm{i}_{\mathbf{u}}\boldsymbol{\omega}\\ &= \mathrm{i}_{\mathbf{u}}\boldsymbol{\omega} - \mathrm{i}_{\mathbf{u}}\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\boldsymbol{\omega}\\ &= \mathrm{i}_{\mathbf{u}}\boldsymbol{\omega}. \end{aligned} \tag{78}$$

To see (12), we check that

$$\begin{aligned}(\mathrm{S}^*_{\mathbf{u}\boldsymbol{\mu}})^{-1}\mathrm{S}^*_{\mathbf{u}\boldsymbol{\mu}}\boldsymbol{\omega} &= \begin{pmatrix}1 & \mathrm{j}_{\boldsymbol{\mu}}\end{pmatrix}\begin{pmatrix}\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\\ \mathrm{i}_{\mathbf{u}}\end{pmatrix}\boldsymbol{\omega}\\ &= (\mathrm{P}^*_{\mathbf{u}\boldsymbol{\mu}} + \mathrm{Q}^*_{\mathbf{u}\boldsymbol{\mu}})\boldsymbol{\omega} = \boldsymbol{\omega}, \end{aligned} \tag{79}$$

for $\boldsymbol{\omega} \in \mathcal{F}^p$, and that

$$\begin{aligned}\mathrm{S}^*_{\mathbf{u}\boldsymbol{\mu}}(\mathrm{S}^*_{\mathbf{u}\boldsymbol{\mu}})^{-1}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} &= \begin{pmatrix}\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\\ \mathrm{i}_{\mathbf{u}}\end{pmatrix}(\boldsymbol{\alpha} + \mathrm{j}_{\boldsymbol{\mu}}\boldsymbol{\beta})\\ &= \begin{pmatrix}(1-\mathrm{j}_{\boldsymbol{\mu}}\mathrm{i}_{\mathbf{u}})\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \end{aligned} \tag{80}$$

for $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{F}^p_{\mathbf{u}} \times \mathcal{F}^{p-1}_{\mathbf{u}}$.

Decomposition of Operators

General decomposition mechanism for operators: Let a general operator $L : \mathcal{F}^p(M) \to \mathcal{F}^q(M)$ be given. If $\boldsymbol{\rho} = L\boldsymbol{\omega}$, the splitting S can be applied to both sides of this relation, and the identity $\mathrm{S}^{-1}\,\mathrm{S}$ may be inserted towards

$$\mathrm{S}\,\boldsymbol{\rho} = \mathrm{S}\,L\,\mathrm{S}^{-1}\,\mathrm{S}\,\boldsymbol{\omega}. \tag{81}$$

Thus, the corresponding operator acting on splitted pairs of differential forms is given by the composition

$$\mathrm{S}\,L\,\mathrm{S}^{-1} : \mathrm{S}\,\mathcal{F}^p = \mathcal{F}^p_{\mathbf{u}} \times \mathcal{F}^{p-1}_{\mathbf{u}} \to \mathcal{F}^q_{\mathbf{u}} \times \mathcal{F}^{q-1}_{\mathbf{u}} = \mathrm{S}\,\mathcal{F}^q. \tag{82}$$

A convenient matrix-vector notation can be used to describe the action of $\mathrm{S}\,L\,\mathrm{S}^{-1}$,

$$(\boldsymbol{\alpha},\boldsymbol{\beta}) \mapsto \begin{pmatrix}L_{q,p} & L_{q,p-1}\\ L_{q-1,p} & L_{q-1,p-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}, \tag{83}$$

with $L_{r,d} : \mathcal{F}^d_{\mathbf{u}} \to \mathcal{F}^r_{\mathbf{u}}$.

Transversal and horizontal derivation: For the horizontal and transversal derivative operators defined by (16) and

$$\mathcal{L}_{\mathbf{u}} = \mathrm{i}_{\mathbf{u}}\mathrm{d} + \mathrm{d}\,\mathrm{i}_{\mathbf{u}}, \tag{84}$$

respectively, we have that

$$\hat{\mathrm{d}}\hat{\mathrm{d}} = -\mathrm{j}_{\boldsymbol{\eta}}\mathcal{L}_{\mathbf{u}}, \tag{85a}$$

$$\hat{\mathrm{d}}\mathcal{L}_{\mathbf{u}} - \mathcal{L}_{\mathbf{u}}\hat{\mathrm{d}} = \mathrm{j}_{\boldsymbol{\delta}}\mathcal{L}_{\mathbf{u}}, \tag{85b}$$

$$\mathcal{L}_{\mathbf{u}}(\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2) = \mathcal{L}_{\mathbf{u}}\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2 + \boldsymbol{\omega}_1 \wedge \mathcal{L}_{\mathbf{u}}\boldsymbol{\omega}_2, \tag{85c}$$

$$\begin{aligned}\hat{\mathrm{d}}\,(\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2) &= \hat{\mathrm{d}}\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2 + (-1)^p\boldsymbol{\omega}_1 \wedge \hat{\mathrm{d}}\boldsymbol{\omega}_2\\ &\quad + \mathrm{i}_{\mathbf{u}}\boldsymbol{\omega}_1 \wedge \boldsymbol{\mu} \wedge \mathrm{d}\boldsymbol{\omega}_2\\ &\quad + (-1)^p\boldsymbol{\mu} \wedge \mathrm{d}\boldsymbol{\omega}_1 \wedge \mathrm{i}_{\mathbf{u}}\boldsymbol{\omega}_2, \end{aligned} \tag{85d}$$

for all $\boldsymbol{\omega}_1 \in \mathcal{F}^p$, $\boldsymbol{\omega}_2 \in \mathcal{F}^q$, and where $\boldsymbol{\eta},\boldsymbol{\delta}$ are given by (18). We note that the second equation states that one may not simply exchange horizontal and transversal differentiation, and that the last equation simplifies to the usual product rule provided that $\boldsymbol{\omega}_1 \in \mathcal{F}^p_{\mathbf{u}}$, $\boldsymbol{\omega}_2 \in \mathcal{F}^q_{\mathbf{u}}$.

Lie derivative, contraction, and multiplication: From the definition (84) of the Lie derivative, it is obvious that

$$\mathrm{i}_{\mathbf{u}}\mathcal{L}_{\mathbf{u}} - \mathcal{L}_{\mathbf{u}}\mathrm{i}_{\mathbf{u}} = 0. \tag{86}$$

For the multiplication, we immediately see that

$$\mathcal{L}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}} - \mathrm{j}_{\boldsymbol{\mu}}\mathcal{L}_{\mathbf{u}} = \mathrm{j}_{\boldsymbol{\delta}}. \tag{87}$$

Combining both (86) and (87) yields

$$[\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}, \mathcal{L}_{\mathbf{u}}] = [\mathcal{L}_{\mathbf{u}}, \mathrm{j}_{\boldsymbol{\mu}}\mathrm{i}_{\mathbf{u}}] = \mathrm{j}_{\boldsymbol{\delta}}\mathrm{i}_{\mathbf{u}}. \tag{88}$$

Exterior derivative: Following the definitions of the splitting, the horizontal exterior derivative, and the Lie derivative, it is easy to see that

$$\mathrm{S}\,\mathrm{d} = \begin{pmatrix}\hat{\mathrm{d}}\\ \mathcal{L}_{\mathbf{u}} - \mathrm{d}\,\mathrm{i}_{\mathbf{u}}\end{pmatrix}. \tag{89}$$

Let $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{F}^p_{\mathbf{u}} \times \mathcal{F}^{p-1}_{\mathbf{u}}$, and use the definitions of the vorticity 2-form and the acceleration 1-form, $\boldsymbol{\eta}$ and $\boldsymbol{\delta}$. With $\mathrm{i}_{\mathbf{u}}\boldsymbol{\alpha} = 0$, $\mathrm{i}_{\mathbf{u}}\boldsymbol{\beta} = 0$, we find

$$\begin{aligned}\mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} &= \mathrm{S}\,\mathrm{d}\,(\boldsymbol{\alpha} + \mathrm{j}_{\boldsymbol{\mu}}\boldsymbol{\beta})\\ &= \begin{pmatrix}\hat{\mathrm{d}}\\ \mathcal{L}_{\mathbf{u}}\end{pmatrix}\boldsymbol{\alpha} + \begin{pmatrix}\hat{\mathrm{d}}\mathrm{j}_{\boldsymbol{\mu}}\\ \mathcal{L}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}} - \mathrm{d}\,\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\end{pmatrix}\boldsymbol{\beta}\\ &= \ldots + \begin{pmatrix}\mathrm{j}_{\boldsymbol{\eta}} - \mathrm{j}_{\boldsymbol{\mu}}\hat{\mathrm{d}} + \mathrm{j}_{\mathrm{i}_{\mathbf{u}}\boldsymbol{\mu}}\mathrm{j}_{\boldsymbol{\mu}}\mathrm{d} - \mathrm{j}_{\boldsymbol{\mu}}\mathrm{j}_{\mathrm{d}\boldsymbol{\mu}}\mathrm{i}_{\mathbf{u}}\\ \mathrm{j}_{\boldsymbol{\delta}} + \mathrm{j}_{\boldsymbol{\mu}}\mathcal{L}_{\mathbf{u}} - \mathrm{d}\,\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\end{pmatrix}\boldsymbol{\beta}\\ &= \ldots + \begin{pmatrix}\mathrm{j}_{\boldsymbol{\eta}} - \mathrm{j}_{\boldsymbol{\mu}}\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\mathrm{d} + \mathrm{j}_{\boldsymbol{\mu}}\mathrm{d}\\ \mathrm{j}_{\boldsymbol{\delta}} + \mathrm{j}_{\boldsymbol{\mu}}\mathrm{i}_{\mathbf{u}}\mathrm{d} - \mathrm{d}\,\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\end{pmatrix}\boldsymbol{\beta}\\ &= \ldots + \begin{pmatrix}\mathrm{j}_{\boldsymbol{\eta}} - \mathrm{j}_{\boldsymbol{\mu}}(1 - \mathrm{j}_{\boldsymbol{\mu}}\mathrm{i}_{\mathbf{u}})\mathrm{d} + \mathrm{j}_{\boldsymbol{\mu}}\mathrm{d}\\ \mathrm{j}_{\boldsymbol{\delta}} + (1 - \mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}})\mathrm{d} - \mathrm{d}\,\mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}}\end{pmatrix}\boldsymbol{\beta}\\ &= \ldots + \begin{pmatrix}\mathrm{j}_{\boldsymbol{\eta}}\\ \mathrm{j}_{\boldsymbol{\delta}} - \hat{\mathrm{d}} + \mathrm{d}(1 - \mathrm{i}_{\mathbf{u}}\mathrm{j}_{\boldsymbol{\mu}})\end{pmatrix}\boldsymbol{\beta}\\ &= \begin{pmatrix}\hat{\mathrm{d}} & \mathrm{j}_{\boldsymbol{\eta}}\\ \mathcal{L}_{\mathbf{u}} & \mathrm{j}_{\boldsymbol{\delta}} - \hat{\mathrm{d}}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}. \end{aligned} \tag{90}$$

Contraction: The splitting S also enables us to decompose the contraction of a p-form $\boldsymbol{\omega} = \boldsymbol{\alpha} + \mathrm{j}_{\boldsymbol{\mu}}\boldsymbol{\beta}$ by a vector field $\mathbf{w} \in \mathcal{X}^1$. We can write $\mathbf{w}$ as

$$\mathbf{w} = \mathbf{a} + b\mathbf{u}, \qquad \mathbf{a} \in \mathcal{X}^1_{\boldsymbol{\mu}},\ b \in \mathcal{X}^0. \tag{91}$$

Observing that

$$\mathrm{i}_{\mathbf{w}}\boldsymbol{\omega} = \mathrm{i}_{\mathbf{a}}\boldsymbol{\alpha} - \mathrm{j}_{\boldsymbol{\mu}}\mathrm{i}_{\mathbf{a}}\boldsymbol{\beta} + b\boldsymbol{\beta}, \tag{92}$$

it is easy to see that the contraction $\mathrm{i}_{\mathbf{w}}$ decomposes to

$$\mathrm{S}\,\mathrm{i}_{\mathbf{w}}\,\mathrm{S}^{-1} = \begin{pmatrix}\mathrm{i}_{\mathbf{a}} & b\\ 0 & -\mathrm{i}_{\mathbf{a}}\end{pmatrix}. \tag{93}$$

Lie Derivative: Let $\mathbf{w} \in \mathcal{X}^1$ be decomposed according to (91) and assume that the splitting $(\mathbf{u},\boldsymbol{\mu})$ is holonomic. We obtain from (19) and (93) that

$$\begin{aligned}&\mathrm{S}\,\mathcal{L}_{\mathbf{w}}\,\mathrm{S}^{-1} = \mathrm{S}\,\mathrm{i}_{\mathbf{w}}\mathrm{S}^{-1}\,\mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1} + \mathrm{S}\,\mathrm{d}\,\mathrm{S}^{-1}\,\mathrm{S}\,\mathrm{i}_{\mathbf{w}}\mathrm{S}^{-1}\\ &= \begin{pmatrix}\mathrm{i}_{\mathbf{a}} & b\\ 0 & -\mathrm{i}_{\mathbf{a}}\end{pmatrix}\begin{pmatrix}\hat{\mathrm{d}} & 0\\ \mathcal{L}_{\mathbf{u}} & -\hat{\mathrm{d}}\end{pmatrix} + \begin{pmatrix}\hat{\mathrm{d}} & 0\\ \mathcal{L}_{\mathbf{u}} & -\hat{\mathrm{d}}\end{pmatrix}\begin{pmatrix}\mathrm{i}_{\mathbf{a}} & b\\ 0 & -\mathrm{i}_{\mathbf{a}}\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{i}_{\mathbf{a}}\hat{\mathrm{d}} + b\mathcal{L}_{\mathbf{u}} & -b\hat{\mathrm{d}}\\ -\mathrm{i}_{\mathbf{a}}\mathcal{L}_{\mathbf{u}} & \mathrm{i}_{\mathbf{a}}\hat{\mathrm{d}}\end{pmatrix} + \begin{pmatrix}\hat{\mathrm{d}}\mathrm{i}_{\mathbf{a}} & \hat{\mathrm{d}}b\\ \mathcal{L}_{\mathbf{u}}\mathrm{i}_{\mathbf{a}} & \mathcal{L}_{\mathbf{u}}b + \hat{\mathrm{d}}\mathrm{i}_{\mathbf{a}}\end{pmatrix}, \end{aligned} \tag{94}$$

with $\widehat{\mathcal{L}}_{\mathbf{a}} = \mathrm{i}_{\mathbf{a}}\hat{\mathrm{d}} + \hat{\mathrm{d}}\mathrm{i}_{\mathbf{a}} : \mathcal{F}^p_{\mathbf{u}} \to \mathcal{F}^p_{\mathbf{u}}$, yielding

$$\mathrm{S}\,\mathcal{L}_{\mathbf{w}}\,\mathrm{S}^{-1} = \begin{pmatrix} \widehat{\mathcal{L}}_{\mathbf{a}} + b\mathcal{L}_{\mathbf{u}} & \mathrm{j}_{\hat{\mathrm{d}}\,b} \\ \mathrm{i}_{\mathcal{L}_{\mathbf{u}}\mathbf{a}} & \widehat{\mathcal{L}}_{\mathbf{a}} + b\mathcal{L}_{\mathbf{u}} + (\mathcal{L}_{\mathbf{u}}\,b) \end{pmatrix}. \tag{95}$$

If furthermore $\mathcal{L}_{\mathbf{u}}\,\mathbf{w} = 0$, we have that

$$\mathcal{L}_{\mathbf{u}}\,b = \mathcal{L}_{\mathbf{u}}\,\mathrm{i}_{\boldsymbol{\mu}}\,\mathbf{w} = \mathrm{i}_{\mathcal{L}_{\mathbf{u}}\boldsymbol{\mu}}\,\mathbf{w} + \mathrm{i}_{\boldsymbol{\mu}}\,\mathcal{L}_{\mathbf{u}}\,\mathbf{w} = 0, \tag{96}$$

since the splitting is assumed to be holonomic and therefore $\boldsymbol{\delta} = \mathcal{L}_{\mathbf{u}}\,\boldsymbol{\mu} = 0$. Since

$$0 = \mathcal{L}_{\mathbf{u}}\,\mathbf{w} = \mathcal{L}_{\mathbf{u}}\,\mathbf{a} + (\mathcal{L}_{\mathbf{u}}\,b)\mathbf{u}, \tag{97}$$

we also have that $\mathcal{L}_{\mathbf{u}}\,\mathbf{a} = 0$. If now additionally $\mathcal{L}_{\mathbf{u}}\,\boldsymbol{\omega} = 0$, we obtain

$$\mathrm{S}\,\mathcal{L}_{\mathbf{w}}\,\mathrm{S}^{-1} = \begin{pmatrix} \widehat{\mathcal{L}}_{\mathbf{a}} & \mathrm{j}_{\hat{\mathrm{d}}\,b} \\ 0 & \widehat{\mathcal{L}}_{\mathbf{a}} \end{pmatrix}. \tag{98}$$

Riesz isomorphism: Employing (3), we have that for $\mathbf{v} \in \mathcal{X}^p$, $\mathbf{w} \in \mathcal{X}^{p-1}$,

$$\begin{aligned} \mathrm{g}\,\mathrm{i}_{\boldsymbol{\mu}}\,\mathbf{v}|\mathbf{w} &= \mathrm{g}\,\mathbf{w}|\mathrm{i}_{\boldsymbol{\mu}}\,\mathbf{v} = \mathrm{j}_{\boldsymbol{\mu}}\,\mathrm{g}\,\mathbf{w}|\mathbf{v} \\ &= \mathrm{g}\,\mathrm{j}_{\mathrm{g}^{-1}\boldsymbol{\mu}}\,\mathbf{w}|\mathbf{v} = \mathrm{g}\,\mathbf{v}|\mathrm{j}_{\mathrm{g}^{-1}\boldsymbol{\mu}}\,\mathbf{w} \\ &= \mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\mu}}\,\mathrm{g}\,\mathbf{v}|\mathbf{w}, \end{aligned} \tag{99}$$

from which follows

$$\mathrm{g}\mathrm{i}_{\boldsymbol{\mu}} = \mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\mu}}\,\mathrm{g}. \tag{100}$$

Relation of $\mathrm{i}_{\mathbf{v}}\boldsymbol{\nu}$, ξ, and η: Applying $\mathbf{g}^{-1}$ to (26) yields

$$\xi^2\mathrm{g}^{-1}\,\boldsymbol{\mu} = \mathbf{u} + \xi\mathbf{v}. \tag{101}$$

Taking the inner product with $\mathbf{u}$ amounts to

$$0 = \langle\mathbf{u},\,\mathbf{u}\rangle + \xi\langle\mathbf{v},\,\mathbf{u}\rangle - \xi^2\langle\mathrm{g}^{-1}\boldsymbol{\mu},\,\mathbf{u}\rangle = \xi\langle\mathbf{v},\,\mathbf{u}\rangle. \tag{102}$$

Therefore, the duality product of (101) with $\boldsymbol{\mu}$ results in

$$\begin{aligned} \xi^2\eta^2 &= \boldsymbol{\mu}|\mathbf{u} + \xi\boldsymbol{\mu}|\mathbf{v} \\ &= 1 + \xi(\xi^{-2}\mathrm{g}\,\mathbf{u} + \xi^{-1}\boldsymbol{\nu})|\mathbf{v} \\ &= 1 + \mathrm{i}_{\mathbf{v}}\,\boldsymbol{\nu}, \end{aligned} \tag{103}$$

which shows (28).

Horizontal volume form: In order to prove (29), we show the defining property

$$(-1)^{\sigma_{\mathrm{o}}}\xi^{-1}\,\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega}|\hat{\mathrm{g}}^{-1}\left((-1)^{\sigma_{\mathrm{o}}}\xi^{-1}\,\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega}\right) = (-1)^{\sigma_{\hat{\mathrm{g}}}}.$$

In particular, we have by the assumption $(-1)^{\sigma_{\hat{\mathrm{g}}}} = (-1)^{(n-1)\sigma_{\mathrm{s}}+\sigma_{\mathrm{g}}}$, as well as by (3), (22), (24), (75), and (100), that

$$\begin{aligned} (-1)^{\sigma_{\hat{\mathrm{g}}}} &= (-1)^{(n-1)\sigma_{\mathrm{s}}+\sigma_{\mathrm{g}}} = (-1)^{(n-1)\sigma_{\mathrm{s}}}\langle\boldsymbol{\Omega},\,\boldsymbol{\Omega}\rangle \\ &= (-1)^{(n-1)\sigma_{\mathrm{s}}}\xi^{-2}\langle\mathbf{u},\,\mathbf{u}\rangle\langle\boldsymbol{\Omega},\,\boldsymbol{\Omega}\rangle \\ &= (-1)^{(n-1)\sigma_{\mathrm{s}}}\xi^{-2}\boldsymbol{\Omega}|\mathrm{g}^{-1}\,((\mathrm{i}_{\mathrm{g}\,\mathbf{u}}\,\mathbf{u})\boldsymbol{\Omega}) \\ &= (-1)^{(n-1)\sigma_{\mathrm{s}}}\xi^{-2}\boldsymbol{\Omega}|\mathrm{g}^{-1}\,(\mathrm{j}_{\mathrm{g}\,\mathbf{u}}\,\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega}) \\ &= (-1)^{(n-1)\sigma_{\mathrm{s}}}\xi^{-2}\boldsymbol{\Omega}|\mathrm{j}_{\mathbf{u}}\,\mathrm{g}^{-1}\,(\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega}) \\ &= (-1)^{(n-1)\sigma_{\mathrm{s}}}\xi^{-2}\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega}|\mathrm{g}^{-1}\,(\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega}) \\ &= \xi^{-2}\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega}|\hat{\mathrm{g}}^{-1}(\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega}), \end{aligned} \tag{104}$$

from which follows the assertion.

Hodge inverse: Let $\boldsymbol{\omega} \in \mathcal{F}^p$ be decomposable, i.e. $\boldsymbol{\omega} = \boldsymbol{\omega}_1 \wedge \ldots \wedge \boldsymbol{\omega}_p$ with $\boldsymbol{\omega}_i \in \mathcal{F}^1$, and set $V = \mathrm{span}\,(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_p)$. Choose $q = n-p$ linear independent 1-forms $\boldsymbol{\omega}_{p+1}, \ldots, \boldsymbol{\omega}_n \in V^{\perp}$ and $\alpha \in \mathcal{F}^0$ such that

$$\boldsymbol{\omega}^{\perp} = \alpha\boldsymbol{\omega}_{p+1} \wedge \ldots \wedge \boldsymbol{\omega}_n \in \mathcal{F}^q \tag{105}$$

satisfies $\boldsymbol{\Omega} = \boldsymbol{\omega} \wedge \boldsymbol{\omega}^{\perp}$. By construction, we have that

$$(\boldsymbol{\omega}^{\perp})^{\perp} = (-1)^{pq}\boldsymbol{\omega}, \tag{106a}$$

$$(\lambda\boldsymbol{\omega})^{\perp} = \lambda^{-1}\boldsymbol{\omega}^{\perp}, \tag{106b}$$

for $\lambda \in \mathcal{F}^0$, $\lambda \neq 0$. Moreover, the definition (30) of the Hodge operator gives

$$\begin{aligned} *\boldsymbol{\omega} &= (-1)^{\sigma_{\mathrm{g}}}\mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\omega}}\,\boldsymbol{\Omega} = (-1)^{\sigma_{\mathrm{g}}}\mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\omega}}\,(\boldsymbol{\omega} \wedge \boldsymbol{\omega}^{\perp}) \\ &= (-1)^{\sigma_{\mathrm{g}}}\langle\boldsymbol{\omega},\,\boldsymbol{\omega}\rangle\,\boldsymbol{\omega}^{\perp}, \end{aligned} \tag{107}$$

where the last step follows from a property of the generalized contraction, [7, eq. (5.58)]. Employing (107), one obtains

$$\langle *\boldsymbol{\omega},\,*\boldsymbol{\omega}\rangle = \langle\boldsymbol{\omega},\,\boldsymbol{\omega}\rangle^2\langle\boldsymbol{\omega}^{\perp},\,\boldsymbol{\omega}^{\perp}\rangle \tag{108}$$

and, therefore,

$$\begin{aligned} (-1)^{\sigma_{\mathrm{g}}} &= \langle\boldsymbol{\Omega},\,\boldsymbol{\Omega}\rangle = \langle\boldsymbol{\omega} \wedge \boldsymbol{\omega}^{\perp},\,\boldsymbol{\omega} \wedge \boldsymbol{\omega}^{\perp}\rangle \\ &= \mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\omega}^{\perp}}\,\mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\omega}}\,(\boldsymbol{\omega} \wedge \boldsymbol{\omega}^{\perp}) \\ &= \langle\boldsymbol{\omega},\,\boldsymbol{\omega}\rangle\langle\boldsymbol{\omega}^{\perp},\,\boldsymbol{\omega}^{\perp}\rangle \\ &= \langle *\boldsymbol{\omega},\,*\boldsymbol{\omega}\rangle\langle\boldsymbol{\omega},\,\boldsymbol{\omega}\rangle^{-1}. \end{aligned} \tag{109}$$

For $\boldsymbol{\omega} \in \mathcal{F}^p$ decomposable, (106)-(107) yield

$$\begin{aligned} **\boldsymbol{\omega} &= (-1)^{\sigma_{\mathrm{g}}}(*\boldsymbol{\omega})^{\perp}\langle *\boldsymbol{\omega},\,*\boldsymbol{\omega}\rangle \\ &= (-1)^{\sigma_{\mathrm{g}}}\left((-1)^{\sigma_{\mathrm{g}}}\langle\boldsymbol{\omega},\,\boldsymbol{\omega}\rangle\,\boldsymbol{\omega}^{\perp}\right)^{\perp}\langle *\boldsymbol{\omega},\,*\boldsymbol{\omega}\rangle \\ &= \langle *\boldsymbol{\omega},\,*\boldsymbol{\omega}\rangle\langle\boldsymbol{\omega},\,\boldsymbol{\omega}\rangle^{-1}(-1)^{pq}\boldsymbol{\omega}. \end{aligned} \tag{110}$$

Together with (109), this gives the fundamental identity

$$**\boldsymbol{\omega} = (-1)^{\sigma_{\mathrm{g}}+p(n-p)}\boldsymbol{\omega}. \tag{111}$$

Since every differential form can be written as the sum of decomposable forms, the identity (111) holds for all $\boldsymbol{\omega} \in \mathcal{F}^p$.

The horizontal Hodge operator: The definition (31) together with (30) and the assumption $(-1)^{\sigma_{\hat{\mathrm{g}}}} = (-1)^{(n-1)\sigma_{\mathrm{s}}+\sigma_{\mathrm{g}}}$ yields (32) by

$$\begin{aligned} \hat{*}\,\boldsymbol{\alpha} &= (-1)^{p\sigma_{\mathrm{s}}+\sigma_{\mathrm{o}}+\sigma_{\hat{\mathrm{g}}}}\xi^{-1}\mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\alpha}}\,\mathrm{i}_{\mathbf{u}}\,\boldsymbol{\Omega} \\ &= (-1)^{p+p\sigma_{\mathrm{s}}+\sigma_{\mathrm{o}}+\sigma_{\hat{\mathrm{g}}}}\xi^{-1}\mathrm{i}_{\mathbf{u}}\,\mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\alpha}}\boldsymbol{\Omega} \\ &= (-1)^{p+p\sigma_{\mathrm{s}}+\sigma_{\mathrm{o}}+\sigma_{\hat{\mathrm{g}}}+\sigma_{\mathrm{g}}}\xi^{-1}\mathrm{i}_{\mathbf{u}}*\,\boldsymbol{\alpha} \\ &= (-1)^{(n-p-1)\sigma_{\mathrm{s}}+\sigma_{\mathrm{o}}}\xi^{-1}\mathrm{i}_{\mathbf{u}}*\mathrm{m}\boldsymbol{\alpha} \\ &= \xi^{-1}\mathrm{n}\,\mathrm{i}_{\mathbf{u}}*\mathrm{m}\boldsymbol{\alpha}, \end{aligned} \tag{112}$$

defining $\mathrm{m}, \mathrm{n} : \mathcal{F}^p \to \mathcal{F}^p$ by (33).

Hodge and contraction: We aim to show the relations

$$*\mathrm{j}_{\boldsymbol{\mu}} = \mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\mu}}*\mathrm{m}, \qquad \mathrm{j}_{\boldsymbol{\mu}}* = -*\mathrm{i}_{\mathrm{g}^{-1}\boldsymbol{\mu}}\mathrm{m}, \tag{113}$$

with m defined by (33). Setting $\mathbf{w} = \mathrm{g}^{-1}\boldsymbol{\mu}$, inserting the definition of the Hodge (30), and employing the properties of the Riesz isomorphism and the contraction operator, we have for $\boldsymbol{\mu} \in \mathcal{F}^1, \boldsymbol{\omega} \in \mathcal{F}^p$

$$\begin{aligned} *\mathrm{j}_{\boldsymbol{\mu}} \boldsymbol{\omega} &= (-1)^{\sigma_{\mathrm{g}}} \mathrm{i}_{\mathrm{g}^{-1} \mathrm{j}_{\boldsymbol{\mu}} \boldsymbol{\omega}} \boldsymbol{\Omega} \\ &= (-1)^{\sigma_{\mathrm{g}}} \mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\omega}} \mathrm{i}_{\mathbf{w}} \boldsymbol{\Omega} \\ &= (-1)^{\sigma_{\mathrm{g}}+p} \mathrm{i}_{\mathbf{w}} \mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\omega}} \boldsymbol{\Omega} \\ &= \mathrm{i}_{\mathbf{w}} * \mathrm{m} \boldsymbol{\omega}, \end{aligned} \tag{114}$$

yielding the first identity of (113). Moreover, employing (111) and (114), we get

$$\begin{aligned} \mathrm{j}_{\boldsymbol{\mu}} * \boldsymbol{\omega} &= (-1)^{(n-p+1)(p-1)+\sigma_{\mathrm{g}}} * * (\mathrm{j}_{\boldsymbol{\mu}} * \boldsymbol{\omega}) \\ &= (-1)^{p(n-p)+p-1+\sigma_{\mathrm{g}}} * (\mathrm{i}_{\mathbf{w}} * * \boldsymbol{\omega}) \\ &= (-1)^{2p(n-p)+p-1} * \mathrm{i}_{\mathbf{w}} \boldsymbol{\omega} \\ &= - * \mathrm{i}_{\mathbf{w}} \mathrm{m} \boldsymbol{\omega}, \end{aligned} \tag{115}$$

which gives the second identity of (113). Moreover, we have that

$$\mathrm{j}_{\boldsymbol{\mu}} \mathrm{i}_{\mathbf{w}} * = \mathrm{j}_{\boldsymbol{\mu}} * \mathrm{j}_{\boldsymbol{\mu}} \mathrm{m} = - * \mathrm{i}_{\mathbf{w}} \mathrm{m} \mathrm{j}_{\boldsymbol{\mu}} \mathrm{m} = * \mathrm{i}_{\mathbf{w}} \mathrm{j}_{\boldsymbol{\mu}} . \tag{116}$$

Decomposition of the Hodge: Considering

$$\begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix} = \mathrm{S} * \mathrm{S}^{-1} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} \mathrm{i}_{\mathbf{u}} \mathrm{j}_{\boldsymbol{\mu}} \\ \mathrm{i}_{\mathbf{u}} \end{pmatrix} * (\boldsymbol{\alpha} + \mathrm{j}_{\boldsymbol{\mu}} \boldsymbol{\beta}), \tag{117}$$

together with (32), (113), and (22), gives

$$\begin{aligned} \boldsymbol{\alpha}' &= \mathrm{i}_{\mathbf{u}} * \mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\mu}} (-\mathrm{m} \boldsymbol{\alpha} + \mathrm{j}_{\boldsymbol{\mu}} \mathrm{m} \boldsymbol{\beta}) \\ &= \mathrm{i}_{\mathbf{u}} * \mathrm{m} (\mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\mu}} \boldsymbol{\alpha} + \mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\mu}} \mathrm{j}_{\boldsymbol{\mu}} \boldsymbol{\beta}) \\ &= \xi \mathrm{n} \hat{*} (\mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\mu}} \boldsymbol{\alpha} + (\eta^2 - \mathrm{j}_{\boldsymbol{\mu}} \mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\mu}}) \boldsymbol{\beta}), \end{aligned} \tag{118}$$

as well as

$$\begin{aligned} \boldsymbol{\beta}' &= \mathrm{i}_{\mathbf{u}} * (\boldsymbol{\alpha} + \mathrm{j}_{\boldsymbol{\mu}} \boldsymbol{\beta}) \\ &= \xi \mathrm{n} \hat{*} (\mathrm{m} \boldsymbol{\alpha} - \mathrm{j}_{\boldsymbol{\mu}} \mathrm{m} \boldsymbol{\beta}). \end{aligned} \tag{119}$$

From (26), we obtain

$$\begin{aligned} \boldsymbol{\mu} &= \xi^{-2} \mathrm{g} \mathbf{u} + \xi^{-1} \boldsymbol{\nu}, \\ \mathrm{g}^{-1} \boldsymbol{\mu} &= \xi^{-2} \mathbf{u} + \xi^{-1} \mathbf{v}, \end{aligned} \tag{120}$$

and inserting that into (118) and (119), together with the fact that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{F}_{\mathbf{u}}^p \times \mathcal{F}_{\mathbf{u}}^{p-1}$, yields (34).

Adapted coordinates: We show that (39) uniquely defines a basis of $\mathcal{X}_{\widetilde{\boldsymbol{\mu}}}^1$ such that the basis is dual to the canonical basis of $\mathcal{F}_{\mathbf{u}}^1$ in adapted coordinates. Let $B_{\boldsymbol{\mu}} = (\partial x_i)$, $i = 1, \ldots, n$, $\partial x_0 = \mathbf{u}$ denote the canonical basis $\mathcal{X}_{\boldsymbol{\mu}}^1$, and $B_{\mathbf{u}}^* = (\mathrm{d}x^i)$ $i = 1, \ldots, n$, $\mathrm{d}x^0 = \boldsymbol{\mu}$ the canoncial basis of $\mathcal{F}_{\mathbf{u}}^1$ in adapted coordinates. Then

$$\mathrm{P}_{\mathbf{u}\boldsymbol{\mu}} : \mathcal{X}_{\widetilde{\boldsymbol{\mu}}}^1 \to \mathcal{X}_{\boldsymbol{\mu}}^1 \tag{121}$$

is an isomorphism with

$$(\mathrm{P}_{\mathbf{u}\boldsymbol{\mu}})^{-1} = \mathrm{P}_{\mathbf{u}\widetilde{\boldsymbol{\mu}}} . \tag{122}$$

Moreover we find

$$\begin{aligned} \delta_j^i &= \mathrm{d}x^i | \partial x_j = \mathrm{d}x^i | \mathrm{P}_{\mathbf{u}\boldsymbol{\mu}} \mathbf{z}_j \\ &= \mathrm{P}_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{d}x^i | \mathbf{z}_j = \mathrm{d}x^i | \mathbf{z}_j . \end{aligned} \tag{123}$$

Hence $B_{\widetilde{\boldsymbol{\mu}}} = (\mathbf{z}_j)$ is the unique basis of $\mathcal{X}_{\widetilde{\boldsymbol{\mu}}}^1$ that is dual to $B_{\mathbf{u}}$. An analog proof holds for $B_{\widetilde{\mathbf{u}}}^* = (\boldsymbol{\zeta}^i)$.

In order to show (44), we set $\mathrm{L}^* = P_{\mathbf{u}\boldsymbol{\mu}}^* P_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}}^* : \mathcal{F}_{\mathbf{u}}^p \to \mathcal{F}_{\mathbf{u}}^p$ and show that

$$\mathrm{L}^* = 1 + \mathrm{j}_{\boldsymbol{\nu}} \mathrm{i}_{\mathbf{v}} . \tag{124}$$

For $P_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}} = \mathrm{i}_{\mathrm{g}\,\mathbf{u}} \mathrm{j}_{\mathrm{g}^{-1} \boldsymbol{\mu}}$, it is obvious that

$$P_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}} \mathbf{u} = \mathbf{u} - \xi^2 \mathrm{g}^{-1} \boldsymbol{\mu}, \tag{125}$$

and that

$$\begin{aligned} P_{\mathbf{u}\boldsymbol{\mu}} P_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}} \mathbf{u} &= \mathrm{i}_{\boldsymbol{\mu}} \mathrm{j}_{\mathbf{u}} P_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}} \mathbf{u} \\ &= -\xi^2 \mathrm{i}_{\boldsymbol{\mu}} \mathrm{j}_{\mathbf{u}} \mathrm{g}^{-1} \boldsymbol{\mu} \\ &= -\xi^2 (\mathrm{g}^{-1} \boldsymbol{\mu} - \eta^2 \mathbf{u}). \end{aligned} \tag{126}$$

Therefore, we can write

$$\begin{aligned} \mathrm{g}^{-1} \boldsymbol{\mu} &= -\xi^{-2} P_{\mathbf{u}\boldsymbol{\mu}} P_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}} \mathbf{u} + \eta^2 \mathbf{u} \\ &= -\xi^{-2} P_{\mathbf{u}\boldsymbol{\mu}} \mathbf{g}^{-1} P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{g} \mathbf{u} + \eta^2 \mathbf{u}. \end{aligned} \tag{127}$$

Using that $P_{\mathbf{u}\boldsymbol{\mu}}^*$ is the identity on $\mathcal{F}_{\mathbf{u}}^p$, and that $P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{j}_{\boldsymbol{\omega}} = \mathrm{j}_{P_{\mathbf{u}\boldsymbol{\mu}}^* \boldsymbol{\omega}} P_{\mathbf{u}\boldsymbol{\mu}}^*$, we have for the operator L^* that

$$\begin{aligned} \mathrm{L}^* &= P_{\mathbf{u}\boldsymbol{\mu}}^* P_{\widetilde{\mathbf{u}}\widetilde{\boldsymbol{\mu}}}^* = P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\mu}} \mathrm{j}_{\mathrm{g}\,\mathbf{u}} \\ &= P_{\mathbf{u}\boldsymbol{\mu}}^* (1 - \mathrm{j}_{\mathrm{g}\,\mathbf{u}} \mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\mu}}) \\ &= 1 - \mathrm{j}_{P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{g}\,\mathbf{u}} P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{i}_{\mathrm{g}^{-1} \boldsymbol{\mu}} . \end{aligned} \tag{128}$$

Inserting (127) and employing $P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{i}_{P_{\mathbf{u}\boldsymbol{\mu}} \mathbf{w}} = \mathrm{i}_{\mathbf{w}} P_{\mathbf{u}\boldsymbol{\mu}}^*$ yield

$$\mathrm{L}^* = 1 + \mathrm{j}_{P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{g}\,\mathbf{u}} (\xi^{-2} \mathrm{i}_{\mathbf{g}^{-1} P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{g}\,\mathbf{u}} P_{\mathbf{u}\boldsymbol{\mu}}^* - P_{\mathbf{u}\boldsymbol{\mu}}^* \eta^2 \mathrm{i}_{\mathbf{u}}), \tag{129}$$

and using again that $P_{\mathbf{u}\boldsymbol{\mu}}^*$ is the identity on $\mathcal{F}_{\mathbf{u}}^p$ and that $\mathrm{i}_{\mathbf{u}} = 0$ on $\mathcal{F}_{\mathbf{u}}^p$ results in

$$\begin{aligned} \mathrm{L}^* &= 1 + \xi^{-2} \mathrm{j}_{P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{g}\,\mathbf{u}} \mathrm{i}_{\mathbf{g}^{-1} P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{g}\,\mathbf{u}} \\ &= 1 + \mathrm{j}_{\xi^{-1} P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{g}\,\mathbf{u}} \mathrm{i}_{\mathbf{g}^{-1} \xi^{-1} P_{\mathbf{u}\boldsymbol{\mu}}^* \mathrm{g}\,\mathbf{u}} . \end{aligned} \tag{130}$$

A comparison with the definition (25) of $\boldsymbol{\nu}$ yields (124).

Moreover, we show that

$$(\mathrm{L}^*)^{-1} = 1 - (\xi\eta)^{-2} \mathrm{j}_{\boldsymbol{\nu}} \mathrm{i}_{\mathbf{v}} : \mathcal{F}_{\mathbf{u}}^p \to \mathcal{F}_{\mathbf{u}}^p, \tag{131}$$

which follows from (28) and

$$\begin{aligned} L^*(L^*)^{-1} &= (1 + \mathrm{j}_{\boldsymbol{\nu}} \mathrm{i}_{\mathbf{v}})(1 - (\xi\eta)^{-2} \mathrm{j}_{\boldsymbol{\nu}} \mathrm{i}_{\mathbf{v}}) \\ &= 1 + (1 - (\xi\eta)^{-2} - \mathrm{i}_{\mathbf{v}} \boldsymbol{\nu} (\xi\eta)^{-2}) \mathrm{j}_{\boldsymbol{\nu}} \mathrm{i}_{\mathbf{v}} \\ &= 1. \end{aligned} \tag{132}$$

With (44) we show (40) using

$$\begin{aligned} \hat{\mathrm{g}}_{ij}^{\dagger} &= \hat{\mathrm{g}}^{\dagger} \mathbf{z}_i | \mathbf{z}_j = (1 + \mathrm{j}_{\boldsymbol{\nu}} \mathrm{i}_{\mathbf{v}}) \hat{\mathrm{g}} \mathbf{z}_i | \mathbf{z}_j \\ &= \hat{g}_{ij} + \mathrm{i}_{\mathbf{v}} \hat{\mathrm{g}} \mathbf{z}_i | \mathrm{i}_{\boldsymbol{\nu}} \mathbf{z}_j \\ &= \hat{g}_{ij} + (-1)^{\sigma_{\mathrm{s}}} \mathrm{g}\, \mathrm{i}_{\boldsymbol{\nu}} \mathbf{z}_i | \mathrm{i}_{\boldsymbol{\nu}} \mathbf{z}_j \\ &= \hat{g}_{ij} + (-1)^{\sigma_{\mathrm{s}}} (\mathrm{i}_{\boldsymbol{\nu}} \mathbf{z}_i)(\mathrm{i}_{\boldsymbol{\nu}} \mathbf{z}_j) \\ &= \hat{g}_{ij} + (-1)^{\sigma_{\mathrm{s}}} \nu_i \nu_j, \end{aligned} \tag{133}$$

where $\hat{\mathrm{g}}_{ij} = \hat{\mathrm{g}}\, \mathbf{z}_i | \mathbf{z}_j$. With $\hat{\mathrm{g}}_{ij}^{\dagger} = (-1)^{\sigma_{\mathrm{s}}} \mathrm{g}_{ij}$, it follows that $\hat{g}_{ij} = (-1)^{\sigma_{\mathrm{s}}} (g_{ij} - \nu_i \nu_j)$.

Since both horizontal volume forms $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}^\dagger$ are elements of the one-dimensional space $\mathcal{F}^{n-1}_{\mathbf{u}}$, we have that $\hat{\mathbf{\Omega}}^\dagger = \alpha\hat{\mathbf{\Omega}}$ for $\alpha \in \mathcal{F}^0$. From that, we get

$$(-1)^{\sigma_{\hat{g}}} = \hat{\mathbf{\Omega}}|\hat{g}^{-1}(\hat{\mathbf{\Omega}}) = \hat{\mathbf{\Omega}}^\dagger|\hat{g}^{\dagger\,-1}(\hat{\mathbf{\Omega}}^\dagger) = \alpha^2\hat{\mathbf{\Omega}}|\hat{g}^{\dagger\,-1}(\hat{\mathbf{\Omega}}). \tag{134}$$

Moreover, it holds that

$$\begin{aligned} j_{\boldsymbol{\nu}}\, i_{\mathbf{v}}\, \hat{\mathbf{\Omega}} &= (-1)^{\sigma_{\hat{g}}} j_{\boldsymbol{\nu}}\, i_{\mathbf{v}}\, \hat{*}1 = (-1)^{\sigma_{\hat{g}}+\sigma_s} j_{\boldsymbol{\nu}}\, \hat{*} j_{\boldsymbol{\nu}}\, 1 \\ &= (-1)^{\sigma_{\hat{g}}}\, \hat{*}\, i_{\mathbf{v}} j_{\boldsymbol{\nu}}\, 1 = (i_{\mathbf{v}}\, \boldsymbol{\nu})\hat{\mathbf{\Omega}}, \end{aligned} \tag{135}$$

and thus that

$$\begin{aligned} \hat{g}^{-1}(\hat{\mathbf{\Omega}}) &= \hat{g}^{\dagger\,-1}\, L^*\, \hat{\mathbf{\Omega}} = \hat{g}^{\dagger\,-1}(1 + j_{\boldsymbol{\nu}}\, i_{\mathbf{v}})\hat{\mathbf{\Omega}} \\ &= \hat{g}^{\dagger\,-1}(1 + i_{\mathbf{v}}\, \boldsymbol{\nu})\hat{\mathbf{\Omega}} = (\xi\eta)^2 \hat{g}^{\dagger\,-1}(\hat{\mathbf{\Omega}}). \end{aligned} \tag{136}$$

Combining (134) and (136), we get $\alpha = \xi\eta$ and thus $\hat{\mathbf{\Omega}}^\dagger = \xi\eta\hat{\mathbf{\Omega}}$.

The relation (45) of the horizontal Hodge and its modification can be concluded from

$$\begin{aligned} \hat{*}\boldsymbol{\omega} &= (-1)^{\sigma_{\hat{g}}}\, i_{\hat{g}^{-1}(\boldsymbol{\omega})}\, \hat{\mathbf{\Omega}} \\ &= (-1)^{\sigma_{\hat{g}}}\, \xi\eta^{-1} i_{\hat{g}^{\dagger\,-1}(1+j_{\boldsymbol{\nu}}\, i_{\mathbf{v}})\boldsymbol{\omega}}\, \hat{\mathbf{\Omega}}^\dagger. \end{aligned} \tag{137}$$

Eventually we derive (46) from (34). With $\hat{*} = (\xi\eta)^{-1}\hat{*}^\dagger(1 + j_{\boldsymbol{\nu}}\, i_{\mathbf{v}})$ we find

$$\hat{*}\, i_{\mathbf{v}} = (\xi\eta)^{-1}\hat{*}^\dagger\, i_{\mathbf{v}} \tag{138}$$

and with (28)

$$\begin{aligned} \hat{*} j_{\boldsymbol{\nu}} &= (\xi\eta)^{-1}\hat{*}^\dagger(1 + j_{\boldsymbol{\nu}}\, i_{\mathbf{v}}) j_{\boldsymbol{\nu}} \\ &= (\xi\eta)^{-1}\hat{*}^\dagger((\xi\eta)^2 - i_{\mathbf{v}}\, j_{\boldsymbol{\nu}}) j_{\boldsymbol{\nu}} \\ &= (\xi\eta)\hat{*}^\dagger j_{\boldsymbol{\nu}}. \end{aligned} \tag{139}$$

Using

$$(1 + j_{\boldsymbol{\nu}}\, i_{\mathbf{v}})(1 - (\xi\eta)^{-2} j_{\boldsymbol{\nu}}\, i_{\mathbf{v}}) = 1 \tag{140}$$

we can rewrite (34) as

$$S * S^{-1} = n\hat{*}^\dagger \begin{pmatrix} (\xi\eta)^{-1} i_{\mathbf{v}} & \eta \\ \eta^{-1}(1 + j_{\boldsymbol{\nu}}\, i_{\mathbf{v}})m & -\xi\eta j_{\boldsymbol{\nu}}\, m \end{pmatrix} \tag{141}$$

from which (46) follows in a straight-forward way.

Electromagnetism in Three Dimensions

Helical symmetry: To investigate helical symmetry we employ helical coordinates (R, ϕ, Z) derived from the cylindrical coordinates (r, φ, z) above via $\phi = \varphi - \alpha z$ for constant α, $R = r$, $Z = z$. The helical metric tensor reads

$$\begin{aligned} g &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & \alpha R^2 \\ 0 & \alpha R^2 & 1 + \alpha^2 R^2 \end{pmatrix}, \qquad (142) \\ g^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 + \frac{1}{R^2} & -\alpha \\ 0 & -\alpha & 1 \end{pmatrix}. \qquad (143) \end{aligned}$$

The volume form is given by $\Omega = R\, dR \wedge d\phi \wedge dZ$.

With $\mathbf{u} = \partial Z = \alpha\partial\varphi + \partial z$ and $\boldsymbol{\mu}^0 = dZ = dz$ we find with $\eta = 1$, $\xi = \sqrt{1 + \alpha^2 R^2}$, $\gamma = \xi^{-1}$

$$\begin{aligned} \mathbf{u}' &= \gamma\mathbf{u}, \qquad (144) \\ \boldsymbol{\mu}' &= \gamma g\, \mathbf{u} = \gamma\left(\alpha R^2 d\phi + \gamma^{-2} dZ\right). \qquad (145) \end{aligned}$$

The splitting S' is anholonomic since

$$d\boldsymbol{\mu}' = 2\gamma\alpha R\, dR \wedge d\phi, \tag{146}$$

and as $\boldsymbol{\delta}' = 0$ we find $\boldsymbol{\eta}' = d\boldsymbol{\mu}'$.

Eventually, with $\widetilde{\mathbf{v}} = \gamma^{-1}\mathbf{u}' - \mathbf{u}^0$, and $\mathbf{u}^0 = \partial z$, we obtain $\widetilde{\mathbf{v}} = \alpha\partial\varphi$.

REFERENCES

[1] F.W. Hehl and Y.N. Obuhov. *Foundations of Classical Electrodynamics - Charge, Flux, and Metric*, volume 33 of *Progress in Mathematical Physics*. Birkhäuser, Boston, Basel, Berlin, 2003.

[2] J. Kocik. Relativistic observer and Maxwell equations: an example of a non principal Ehresmann connection. Preprint P-98-10-029, Department of Physics, UIUC, Urbana, IL61801, 1998.

[3] R. T. Jantzen, P. Carini, and D. Bini. GEM: the user manual - understanding spacetime splittings and their relationships, 2004.

[4] M. Fecko. On 3+1 decomposition with respect to an observer field via differential forms. *J. Math. Phys.*, 38(9), 1997.

[5] L. I. Schiff. A question in general relativity. *Proceedings of the National Academy of Sciences of the United States of America*, 25(7):391–395, June 1939.

[6] A. Bossavit. Extrusion, contraction: their discretization via Whitney forms. *COMPEL*, 22(3):470–480, 2003.

[7] W. H. Greub. *Multilinear Algebra*, volume XII of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin/Heidelberg/New York, 1967.

[8] W. Burke. *Applied Differential Geometry*. Cambridge University Press, Cambridge, UK, 1985.

[9] J. van Bladel. *Relativity and Engineering*. Springer, 1984.

[10] B. Mashhoon. The hypothesis of locality in relativistic physics. *Physics Letters A*, 145(4):147–153, April 1990.

[11] J. van Bladel. Electromagnetic fields in the presence of rotating bodies. *Proceedings of the IEEE*, 64(3):301–318, March 1976.

[12] T.C. Mo. Theory of electrodynamics in media in noninertial frames and applications. *Journal of Mathematical Physics*, 11(8):2589–2610, 1970.

[13] A. Nicolet and F. Zolla. Finite element analysis of helicoidal waveguides. *IET Sci. Meas. Technol.*, 1(1):67–70, January 2007.