Nonconforming discretization techniques for overlapping domain decompositions

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Summary. For the numerical solution of coupled problems on two nested domains, two meshes are used which are completely independent to each other. Especially in the case of a moving subdomain, this leads to a great flexibility for employing different meshsizes, discretizations or model equations on the two domains. We present a general setting for these problems in terms of saddle point formulations, and investigate one- and bi-directionally coupled applications.

1 Introduction

We consider coupled problems on two nested domains, the global domain Ω and the subdomain ω, see Figure 1. In order to approximate the involved solution components on Ω and ω, two meshes are used which are completely independent to each other. We

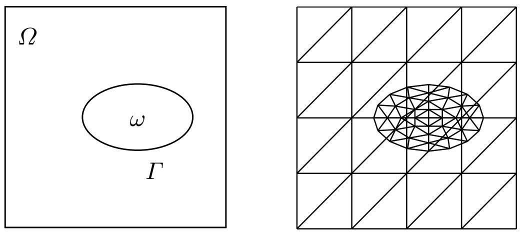

Fig. 1. Two nested domains (left), independent grids (right).

like to be able to deal with different meshsizes, discretizations and model equations on the two domains. Our approach is useful especially for a moving subdomain, i.e., when ω changes its position inside the global domain. In this case, no remeshing will be necessary and only the matrices responsible for the coupling have to be reassembled. In Section 2, we start with the general variational setting in terms of a saddle point formulation. A one-directionally coupled model problem is investigated in Section 3. In Section 4, we consider bi-directionally coupled formulations on the examples of a linear elasticity problem and an eddy current simulation.

2 Variational Setting

Generalized saddle point problems. Our goal is to find a solution $u = (u_\Omega, u_\omega)$ consisting of two components defined on the global domain Ω and on the subdomain ω, respectively. We denote by V_Ω and V_ω the appropriate weak function spaces for the solution components as well as for the test functions. Without taking into account any coupling between the two solution components, the involved differential operators are in general described by continuous bilinear forms $a_\Omega(\cdot,\cdot)$ acting on $V_\Omega \times V_\Omega$, and $a_\omega(\cdot,\cdot)$ acting on $V_\omega \times V_\omega$. Indicating by V the product space $V_\Omega \times V_\omega$, a composed bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is obtained by

$$a(w,v) = a_\Omega(w_\Omega, v_\Omega) + a_\omega(w_\omega, v_\omega), \quad w, v \in V.$$

The coupling between the two solution components is realized via the Lagrange multiplier space M in terms of two continuous bilinear forms $b_1(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$ acting on $V \times M$. For the applications in Section 3 and Subsection 4.2, M is the dual of the trace space $H^{1/2}(\Gamma)$, i.e., $M = H^{-1/2}(\Gamma)$, whereas in Subsection 4.1 $M = H^{-1/2}(\Gamma)^2$, with $\Gamma = \partial\omega$ indicating the subdomain boundary.

Solving additionally for the Lagrange multiplier $p \in M$, the following generalized saddle point problem is derived: find $(u,p) \in V \times M$ such that

$$\begin{aligned} a(u,v) + b_1(v,p) &= \langle f, v\rangle_{V' \times V}, && v \in V, \\ b_2(u,q) &= \langle g, q\rangle_{M' \times M}, && q \in M, \end{aligned} \tag{1}$$

where $\langle\cdot,\cdot\rangle_{V'\times V}$ and $\langle\cdot,\cdot\rangle_{M'\times M}$ denote the usual duality pairings. We point out that for $b_1(\cdot,\cdot) = b_2(\cdot,\cdot)$, problem (1) has the usual symmetric structure, which is encountered for example in the framework of mixed [1] and mortar [2] finite element methods. Moreover, if $b_1(\cdot,q)$ acts only either on V_Ω or V_ω, one-directionally coupled problems are derived.

The bilinear forms $b_i(\cdot,\cdot)$ define coupling operators $B_i : V \to M'$ and $B_i^{\mathrm{T}} : M \to V'$ by $\langle B_i v, q\rangle_{M'\times M} = \langle v, B_i^{\mathrm{T}} q\rangle_{V\times V'} = b_i(v,q)$ for $v \in V$ and $q \in M$. The validation of the following coercivity- and inf-sup-conditions guarantees the unique solvability of problem (1) in $V \times M/KerB_1^{\mathrm{T}}$, [5]:

$$\exists \alpha_0 > 0 : \sup_{v_0 \in \operatorname{Ker} B_1} \frac{a(w_0,v_0)}{\|w_0\|_V \|v_0\|_V} \ge \alpha_0, \quad w_0 \in \operatorname{Ker} B_2, \tag{2}$$

$$\sup_{w_0 \in \operatorname{Ker} B_2} \frac{a(w_0,v_0)}{\|w_0\|_V \|v_0\|_V} \ge \alpha_0, \quad v_0 \in \operatorname{Ker} B_1. \tag{3}$$

$$\exists k_0 > 0 : \inf_{q\in M} \sup_{v \in V} \frac{b_i(v,q)}{\|v\|_V \|q\|_{M/\operatorname{Ker} B_i^{\mathrm{T}}}} \ge k_0, \quad i = 1,2. \tag{4}$$

We note that the above conditions can be more relaxed [3, 11].

Discretization. We use two different shape regular quasi-uniform triangulations $\mathcal{T}_H$ on Ω and $\mathcal{T}_h$ on ω, as illustrated in Figure 1, with H and h indicating the corresponding maximum element diameter. The function spaces V_Ω, V_ω, and M are replaced by discrete approximations $V_H \subset V_\Omega$, $V_h \subset V_\omega$, and $M_h \subset M$, respectively. We denote an element (v_H, v_h) of the product space $V_h^H = V_H \times V_h$ by v_h^H. It may become necessary to involve approximate bilinear forms $a_h(\cdot,\cdot)$, $b_{1,h}(\cdot,\cdot)$, and $b_{2,h}(\cdot,\cdot)$. The discrete saddle point formulation reads: find $(u_h^H, p_h) \in V_h^H \times M_h$ such that

$$\begin{aligned} a_h(u_h^H, v) + b_{1,h}(v, p_h) &= \langle f, v\rangle_{V'\times V}, && v \in V_h^H, \\ b_{2,h}(u_h^H, q) &= \langle g, q\rangle_{M'\times M}, && q \in M_h, \end{aligned} \tag{5}$$

If the conditions corresponding to (2)–(4) hold for $a_h(\cdot,\cdot)$, $b_{1,h}(\cdot,\cdot)$, and $b_{2,h}(\cdot,\cdot)$ with constants independent of the meshsizes H and h, it is possible to derive optimal a priori estimates. This is a consequence of the next lemma, which follows from [3, Thm. 2.2].

Lemma 1. *Under conditions (2)–(4), the following estimate holds with a constant C depending on α_0, k_0 and the continuity constants of the involved bilinear forms:*

$$\|u-u_h^H\|_V + \|p-p_h\|_M \le C \inf_{v\in V_h^H}\|u-v\|_V + C\inf_{q\in M_h}\|p-q\|_M$$
$$+C\sup_{v\in V_h^H}\frac{(a-a_h)(u,v)}{\|v\|_V} + C\sup_{v\in V_h^H}\frac{(b_1-b_{1,h})(v,p)}{\|v\|_V} + C\sup_{q\in M_h}\frac{(b_2-b_{2,h})(u,q)}{\|q\|_M}.$$

The most delicate step for the quality of the discretization and the computational complexity is the information transfer between the two grids via the discrete Lagrange multiplier space M_h. In all our considered applications, we essentially couple between the global grid on Ω and the subdomain boundary Γ. On Γ, dual Lagrange multipliers [13] are used to approximate M, which have optimal stability and approximation properties. Moreover, they have local support and satisfy a biorthogonality relation with the basis functions of the trace space $V_{h|\Gamma}$. Therefore, the implementation of the corresponding operators $B_{1,h}$ and $B_{2,h}$ can be performed with low computational costs.

3 A one-directionally coupled model problem

We apply the framework presented in the last section to a one-directionally coupled model problem. We present a uniqueness proof and an a priori error estimate, which we confirm by a numerical example.

Continuous formulation. Consider the problem

$$-\Delta u_\Omega = f_\Omega \text{ in } \Omega, \quad u_{\Omega|\partial\Omega} = 0, \tag{6}$$

with its associated bilinear form $a_\Omega(w_\Omega, v_\Omega) := \int_\Omega \nabla w_\Omega \nabla v_\Omega$, for $w_\Omega, v_\Omega \in H_0^1(\Omega)$. We want to solve an additional problem for the subdomain ω, namely,

$$-\Delta u_\omega = f_\omega \text{ in } \omega, \quad u_{\omega|\Gamma} = u_{\Omega|\Gamma}, \tag{7}$$

with $f_\omega = f_{\Omega|\omega}$. Using Green's formula, we obtain

$$a_\omega(u_\omega, v_\omega) + \langle v_\omega, \frac{\partial u_\omega}{\partial n}\rangle_{M'\times M} = (f_\omega, v_\omega)_\omega, \quad v_\omega \in H^1(\omega),$$

with the obvious meanings for $a_\omega(\cdot,\cdot)$ and $(\cdot,\cdot)_\omega$, and where $M = H^{-1/2}(\Gamma)$. Introducing the Lagrange multiplier $p = \frac{\partial u_\omega}{\partial n}$, we find that $b_1(v,q) = \langle v_\omega, q\rangle_{M'\times M}$ for $v = (v_\Omega, v_\omega) \in V = H_0^1(\Omega)\times H^1(\omega)$, $q \in M$. We realize the continuity requirement along the boundary Γ in (7) by the bilinear form $b_2(v,q) = \langle v_\Omega - v_\omega, q\rangle_{M'\times M}$, $v \in V$, $q \in M$ and obtain the saddle point formulation of (6), (7) given in (1) with $g = 0$.

It is obvious that problem (1) has a unique solution, since the problem on the global domain Ω is not influenced by the problem on the subdomain ω, and its solution u_Ω yields the boundary data for a well posed subdomain problem. Nevertheless, we provide a complete proof within the saddle point setting.

Theorem 1. *With the above definitions, problem (1) is uniquely solvable.*

Proof. We first show the unique solvability of problem (1) by validating the conditions (2)–(4). Our main tool is the harmonic extension operator $\mathcal{H} : M' \to V_\omega$, defined by

$$a_\omega(\mathcal{H}w, v_\omega) = 0, \quad v_\omega \in V_\omega^0 = H_0^1(\omega), \quad (\mathcal{H}w)_{|\Gamma} = w. \tag{8}$$

We observe that the trace of V_ω onto Γ is the space M, and that $b_1((0,v),q) = b_2((0,-v),q)$. Taking $v = (0, \pm\mathcal{H}w)$, $w \in M'$, condition (4) is a consequence of the definition of the $H^{-1/2}$-norm and of the fact that $\|\mathcal{H}w\|_{1,\omega} \le C\|w\|_{M'}$.

Let us focus on condition (2). The kernels of the coupling operators are $\operatorname{Ker} B_1 = V_\Omega \times V_\omega^0$, and $\operatorname{Ker} B_2 = \{v \in V : \operatorname{tr} v_\Omega = v_{\omega|\Gamma}\}$, where $\operatorname{tr} : H^1(\Omega) \to H^{1/2}(\Gamma)$ denotes the trace operator. We uniquely decompose $v_\omega \in V_\omega$ into $v_B + v_I$ such that $v_B = \mathcal{H}(v_{\omega|\Gamma})$ and $v_I \in V_\omega^0$. For an arbitrary $w_0 = (w_\Omega, w_B + w_I) \in \operatorname{Ker} B_2$, we consider $v_0 = (w_\Omega, w_I) \in \operatorname{Ker} B_1$. By using the properties of the harmonic extension, we get

$$\|w_\Omega\|_{1,\Omega}^2 \ge c\left(\int_\Gamma \operatorname{tr} w_\Omega\right)^2 + c|w_\Omega|_{1,\Omega}^2 \ge c\left(\int_\Gamma w_B\right)^2 + c|w_B|_{1,\omega}^2 \ge c\|w_B\|_{1,\omega}^2. \tag{9}$$

Condition (2) follows from (9):

$$a(w_0, v_0) = a_\Omega(w_\Omega, w_\Omega) + a_\omega(w_I, w_I) \ge c\|w_\Omega\|_{1,\Omega}^2 + c\|w_I\|_{1,\omega}^2 \ge c\|w_0\|_1\|v_0\|_1. \tag{10}$$

The proof of condition (3) is similar. For an arbitrary $v_0 = (v_\Omega, v_I) \in \operatorname{Ker} B_1$, we set $w_0 = (v_\Omega, \mathcal{H}(\operatorname{tr} v_\Omega) + v_I) \in \operatorname{Ker} B_2$, and obtain (10).

Discretization. We use standard conforming finite elements of order r and s on $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively. The associated discrete spaces with no boundary conditions are denoted by $S_H^r(\Omega)$ and $S_h^s(\omega)$, and we set $S_{0,H}^r(\Omega) = S_H^r(\Omega) \cap H_0^1(\Omega)$ and $S_{0,h}^s(\omega) = S_h^s(\omega) \cap H_0^1(\omega)$ to be the spaces taking into account homogeneous Dirichlet conditions on $\partial\Omega$ and Γ, respectively. For the discrete Lagrange multiplier space M_h, we propose the use of dual basis functions [13] adapted to the order s of elements from the trace space of $S_h^s(\omega)$, which is indicated by $W_h^s(\Gamma)$. Setting $V_h^H = S_{0,H}^r(\Omega) \times S_h^s(\omega)$, the discrete saddle point problem (5) is obtained. The unique solvability of problem (5) can be shown by replacing the harmonic extension $\mathcal{H}$ and the trace operator tr in the proof of Theorem 1 by discrete operators $\mathcal{H}_h : W_h^s(\Gamma) \to S_h^s(\omega)$ and $\operatorname{tr}_h : S_{0,H}^r(\Omega) \to W_h^s(\Gamma)$, respectively. In order to obtain the estimate (9), these operators have to satisfy certain stability and extension properties with respect to the $H^{1/2}$-norm. The discrete harmonic extension $\mathcal{H}_h$ is naturally obtained by taking $S_{0,h}^s(\omega)$ as a test function space in (8), and the operator tr_h is given by the mortar projection associated with the discrete Lagrange multiplier space M_h, in particular, for this choice, we find $\int_\Gamma w_\Omega = \int_\Gamma \operatorname{tr}_h w_\Omega$.

We intend to use a smaller meshsize $h < H$ or a higher order $s > r$ on the subdomain, and, therefore, expect a better solution u_h compared to $u_{H|\omega}$. Thus, the finite element solution u_{FE} is defined by

$$u_{\mathrm{FE}} := \begin{cases} u_H & \text{in } \omega^c = \Omega \setminus \overline{\omega}, \\ u_h & \text{in } \omega. \end{cases}$$

Lemma 1 only provides a global estimate, which is not sufficient here, since we like to disregard the approximate solution component u_H on the subdomain ω. The necessary tools for a more local analysis can be found in [12], resulting in the following estimate which is proved in [7].

Theorem 2. *Let $B \supset \omega^c$ such that $d = \operatorname{dist}(\partial B \setminus \partial\Omega, \partial\omega^c \setminus \partial\Omega) > 0$. Then for H small enough and u regular enough, there exists a constant C depending on d such that*

$$\|u - u_H\|_{1,\omega^c} + \|u - u_h\|_{1,\omega} \le Ch^s|u|_{s+1,\omega} + CH^r|u|_{r+1,B} + CH^{r+1}|u|_{r+1,\Omega}. \tag{11}$$

We note that the last term in (11) is the fundamental difference of our approach to the estimates obtained by standard adaptive finite element methods. It is due to the fact that in our one-directionally coupled approach no pollution effect is taken into account.

Numerical test. Consider the model problem (6) on $\Omega := (0,1)^2$ with source term f derived from the exact solution $u(x,y) := \exp(-100((x-0.6)^2/a^2 + (y-0.5)^2/b^2))$. An elliptic patch with radii 0.25 and 0.15 is placed in the domain Ω with its center at $(0.6, 0.5)$, as illustrated in Figure 1. Since the solution goes to zero with an exponential decay, we may have a coarser triangulation far enough away from $(0.6, 0.5)$. Therefore, we choose an initial triangulation with $h/H = 1/4$. We use $P1$ elements on $\mathcal{T}_H$, whereas on $\mathcal{T}_h$, we consider two different cases and use $P1$ elements for one test, and $P2$ elements for another test. Figure 2 shows the decay of the errors $e_H = u - u_H$ and $e_{\mathrm{FE}} = u - u_{\mathrm{FE}}$ in the H^1-norm under uniform refinement. The errors e_H and e_{FE} both satisfy the a priori

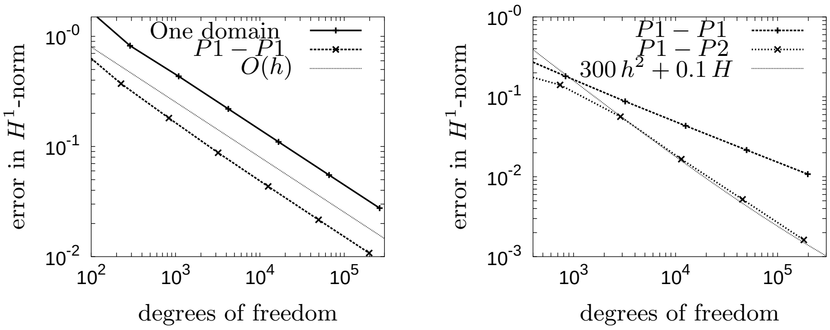

Fig. 2. Error decay in the H^1-norm of $P1$-$P1$ and $P1$-$P2$ coupling

estimates. Choosing the same number of unknowns for the standard and the overlapping method, the solution obtained by the $P1$-$P1$ coupling is significantly better than the solution obtained by the standard method. For the $P2$-$P1$ coupling, the error decay is almost optimal with respect to the piecewise quadratic finite elements used on $\mathcal{T}_h$. In agreement with (11), the error behaves like $c_1 h^2 + c_2 H$, and, numerically, $c_2 \ll c_1$.

4 Bi-directionally coupled problems

We present two applications which result in bi-directionally coupled problems. The first one illustrates a complementary coupling procedure, the second one considers an eddy current problem.

4.1 Natural boundary conditions at the hole ω

We want to solve a boundary value problem on the domain $\Omega \setminus \overline{\omega} =: \omega^c$ with natural boundary conditions on the hole boundary Γ. The solution on the global domain Ω yields the solution on the domain with hole ω^c. This problem is analyzed for the linear elasticity setting in [9]. In addition, we show an application for rotating "holes", see Figure 4.

Saddle point formulation for the linear elasticity problem. We consider the problem: find $u_c \in \left(H^1(\omega^c)\right)^2$ such that

$$\begin{aligned} -\mathsf{Div}\,\sigma(u_c) &= f_c && \text{in } \omega^c, \\ \sigma(u_c)\,n_c &= t && \text{on } \Gamma, \end{aligned}$$

with $u_c = 0$ on Γ_0 and $\sigma(u_c)\,n_c = g$ on Γ_1 where $\Gamma_0 \subset \partial\Omega$ has a positive measure and $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, with body forces $f_c \in \left(L^2(\omega^c)\right)^2$ and surface tractions $g \in \left(L^2(\Gamma_1)\right)^2$, $t \in \left(L^2(\Gamma)\right)^2$. An equivalent formulation is given by

$$
\begin{aligned}
-\mathsf{Div}\,\sigma(u_\Omega) &= f_\Omega && \text{in } \Omega,\\
-\mathsf{Div}\,\sigma(u_\omega) &= f_\omega && \text{in } \omega,\\
u_\Omega &= u_\omega && \text{on } \Gamma,\\
\big[\sigma(u_\Omega)\,n_c\big] &= -\sigma(u_\omega)\,n_c + t && \text{on } \Gamma,
\end{aligned}
$$

with $u_\Omega = 0$ on Γ_0 and $\sigma(u_\Omega)\,n_c = g$ on Γ_1, and where $[w_\Omega] := w_{\Omega|_{\omega^c}} - w_{\Omega|_\omega}$, $f_\omega = f_{\Omega|_\omega}$ and $f_\Omega \in \big(L^2(\Omega)\big)^2$ is an extension of f_c to Ω. It can be written in its weak formulation as saddle point problem with the bilinear forms $a_\Omega(u_\Omega, v_\Omega) = \int_\Omega \sigma(u_\Omega) : \epsilon(v_\Omega)$, $a_\omega(u_\omega, v_\omega) = \int_\omega \sigma(u_\omega) : \epsilon(v_\omega)$, $b_1(v,q) = \langle v_\omega + v_\Omega,\, q\rangle_{M\times M'}$, $b_2(v,q) = \langle v_\omega - v_\Omega,\, q\rangle_{M\times M'}$, and $V = \big(H^1(\Omega)\big)^2 \times \big(H^1(\omega)\big)^2$, $M = \Big(H^{-\frac{1}{2}}(\Gamma)\Big)^2$. For the discretization, we use linear and quadratic finite elements on quasi-uniform and shape regular triangulations. The conditions (2) – (4) hold for h/H small enough in the discrete setting as well as in the continuous setting, yielding unique solvability. For details, we refer to [9].

The realization of this approach allows for an easy shift of the hole without having to remesh and can be used in shape optimization algorithms to determine an optimal hole position. Note that the quantity we pass back from the hole to the background is the jump in the fluxes, i.e., in general the solution u_Ω is only $H^{\frac{3}{2}-\varepsilon}$-regular, $\varepsilon > 0$. Another application of this complementary coupling technique are time dependent problems where the hole is an object moving through the domain Ω emanating some flux into ω^c.

Numerical examples

Beam with one hole. We consider the problem domain $\omega^c = (-5,5)\times(0,1)\setminus\big\{(x,y) \in \mathbf{R}^2 \mid \|(x,y) - (-1,0.5)\| \le 0.3\big\}$ with $u_c = 0$ for $x = \pm 5$, $\sigma(u_c)\,n_c = (0,-1)$ for $y = 1$, $\sigma(u_c)\,n_c = 0$ elsewhere and $f = 0$. We use Young's modulus $E = 200$ and Poisson ratio $\nu = 0.3$. The stress σ_{xx} is monitored as a graphical quantity, see Figures 3. The iterative solver is based on a block Gauß-Seidel method for the symmetric positive definite system arising from static condensation of the Lagrange multiplier. The convergence rates are level independent, see [9].

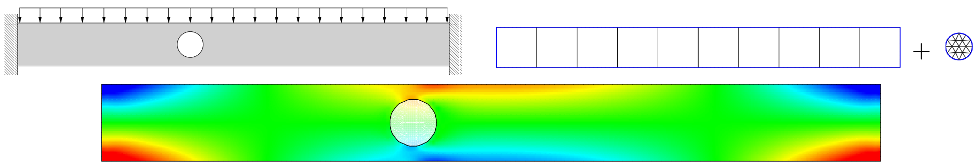

Fig. 3. Top: Problem setup and start grids. Bottom: σ_{xx} on $\Omega\times\omega$ using complementary coupling; we show only the values on ω^c

Rotating hole. The hole domain is a smooth star with five points described by $(x,y) = (m_x, m_y) + (r_i + \Delta r)\cos(10\pi\lambda)\,(\cos(\alpha(\lambda,t)), \sin(\alpha(\lambda,t)))$ with $\lambda, t \in [0,1]$ and $\alpha(\lambda,t) = 2\pi(\lambda - t - cos(10\pi(\lambda + \frac{1}{8})/51))$, and a center $(m_x, m_y) = (1.1, 0.65))$, the medium radius $r_i = 0.2$ and the radius change amplitude $\Delta r = 0.05$. We solve the Poisson problem. The boundary segment described by $\lambda \in (0, 0.1)$ carries non-zero natural data. The background region is $\Omega = (0,3)\times(0,1)$. The situation and the solution at $t = 0, 0.15, 0.3, 0.45, 0.65, 0.85$ can be found in Figure 4. For each new position of the star, no remeshing procedure has to be carried out, simply the existent grid has to be rotated. Moreover, only the coupling matrix has to be reassembled, all other involved matrices stay the same throughout the whole computation.

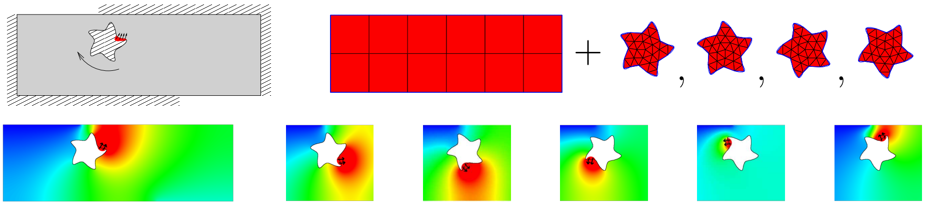

Fig. 4. Problem setup (top left): Zero natural b.c. in hatched areas, 0 and 1 Dirichlet b.c. at the top left and bottom right side, respectively, influx of 5 (natural b.c.) at the back side of the first wing. Initial background grid plus rotated star grid at $t = 0, 0.15, 0.3, 0.45$ (top right). Bottom: Solutions at different times t: $t = 0$ (complete), $t = 0.15, 0.3, 0.45, 0.65, 0.85$ (partly displayed).

4.2 Eddy current simulation

We want to approximate the eddy currents inside a conductor ω which is exposed to a time dependent electromagnetic field acting in the global domain Ω. A detailed problem description and analysis concerning the statically condensated elliptic system can be found in [8], numerical results are available in [6]. Here, we present an alternative approach which fits into the saddle point framework presented in Section 2.

Saddle point formulation. Elimination of the other involved field quantities from the quasistationary Maxwell equations yields for the magnetic field H

$$\operatorname{div} \mu H = 0 \quad \text{in } \Omega, \tag{12}$$

$$\partial_t H + \frac{1}{\sigma\mu} \operatorname{curl} \operatorname{curl} H = 0 \quad \text{in } \omega, \tag{13}$$

with positive material parameters μ and σ, and μ constant in ω. We assume knowing a source vector potential T_s such that $\operatorname{curl} T_s = J_s$ in ω^c, with J_s a given source current density. The magnetic field H is decomposed into $T - \operatorname{grad}\phi$ on ω and $T_s - \operatorname{grad}\phi$ on ω^c, where $T \in H_0^{\text{curl}}(\omega)$ is a vector valued potential defined on the conductor ω, and $\phi \in H_0^1(\Omega)$ is a scalar valued potential defined on the global domain Ω. Moreover, we use the Coulomb gauge, i.e., T is chosen to be solenoidal. From (12), we obtain

$$a_\Omega(\phi, v) - \int_\Gamma (Tn)v = \int_{\omega^c} \beta T_s \operatorname{grad} v, \quad v \in V_\Omega = H_0^1(\Omega), \tag{14}$$

with $a_\Omega(w, v) = \int_\Omega \beta \operatorname{grad} w \operatorname{grad} v$ and β depending on μ. Taking $v \in H_0^1(\omega)$, (14) implies that ϕ is harmonic on ω, thus, there exists $\gamma = \phi_{|\Gamma} \in H^{1/2}(\Gamma)$ such that $\phi_{|\omega} = \mathcal{H}\gamma$ with the harmonic extension operator $\mathcal{H} : H^{1/2}(\Gamma) \to H^1(\omega)$. Furthermore, due to the solenoidality of T, it holds that $\int_\Gamma (Tn)\lambda - \int_\omega T \operatorname{grad} \mathcal{H}\lambda = 0$ for an arbitrary $\lambda \in H^{1/2}(\Gamma)$. After time discretization by an implicit Euler scheme with time step size Δt, we obtain from (13) at each time step:

$$a_\omega((\gamma, T), (\lambda, W)) + \int_\Gamma (Tn)\lambda = \int_\omega f_\omega W, \quad (\lambda, W) \in V_\omega = H^{1/2}(\Gamma) \times H_0^{\text{curl}}(\omega), \tag{15}$$

with

$$a_\omega((\gamma, T), (\lambda, W)) = \int_\omega \alpha \operatorname{curl} T \operatorname{curl} W + TW - W \operatorname{grad} \mathcal{H}\gamma - T \operatorname{grad} \mathcal{H}\lambda, \tag{16}$$

where $\alpha = \Delta t/(\mu\sigma)$, and f_ω contains the information from the preceding time step.

This suggests the introduction of the Lagrange multiplier $p = Tn \in M = H^{-1/2}(\Gamma)$, and of the coupling bilinear form $b((v,\lambda,W),q) = \langle \lambda - v, q\rangle_{M'\times M}$ for $(v,\lambda,W) \in V = V_\Omega \times V_\omega$ and $q \in M$. Setting $g = 0$ and $b_1(\cdot,\cdot) = b_2(\cdot,\cdot) = b(\cdot,\cdot)$, problem (1) is obtained. By choosing $(\lambda, W) = (0, \operatorname{grad} v)$, $v \in H_0^1(\omega)$, in (15), it is easy to see that the solenoidality of T is guaranteed provided that f_ω is divergence free. The unique solvability of the statically condensated formulation of problem (1) is proved in [8].

Discretization. For the approximation of ϕ, piecewise linear finite elements are used on $\mathcal{T}_H$. Concerning the vector potential T, we employ curl-conforming edge elements [10] on $\mathcal{T}_h$, which are ideally suited for the approximation of $H_0^{\text{curl}}(\omega)$, whereas for γ, we use piecewise linear finite elements on Γ. As before, we approximate the Lagrange multiplier space M by dual basis functions. In contrast to the preceding applications, the bilinear form $a(\cdot,\cdot)$ cannot be implemented directly, and we need an approximation $a_h(\cdot,\cdot)$. Therefore, the harmonic extension operator $\mathcal{H}$ in (16) is replaced by its discrete analogue $\mathcal{H}_h$ corresponding to piecewise linear finite elements on $\mathcal{T}_h$. The gradient operator can be easily realized by the node-to-edge incidence matrix G which acts on the degrees of freedom associated with the linear elements on $\mathcal{T}_h$, and gives the gradient as a linear combination of the basis functions for the edge element space [4]. An optimal a priori estimate based on Lemma 1 for the finite element solution (ϕ_H, T_h) of the statically condensated form of (5) is obtained in [8], provided that the ratio h/H is small enough.

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